A forest-fire model and some thoughts on turbulence

Per Bak, Kan Chen
Department of Physics, Brookhaven National Laboratory, Upton, NY 11973, USA

and

Chao Tang
Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA

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In the context of a forest-fire model we demonstrate critical scaling behavior in a "turbulent" non-equilibrium system. Energy is injected uniformly, and dissipated on a fractal. Critical exponents are estimated by means of a Monte Carlo renormalization-group calculation.

Many extended dissipative systems exhibit spatial and temporal scaling. In turbulence, for example, scaling occurs over wide length scales since the coherence length is usually much larger than the Kolmogorov length (the length scale for energy dissipation). Uniform energy input results in power-law spatial distribution of energy storage (Kolmogorov scaling) and fractal energy dissipation [1]. Some phenomenological models for scaling behaviors have been proposed [2], in which the fractal set is preassumed. It is, however, essential to understand the dynamical mechanism which generates the fractal itself: How can a uniform energy injection result in a fractal dissipation?

Motivated by the recent discovery of self-organized criticality [3] we speculate that fractal energy dissipation is a manifestation of a critical state. Our study of fractal energy dissipation is not in the context of turbulent fluids, but as a general phenomenon. The model we study is a simple "forest-fire" model, and we focus on the spatial distribution of dissipation (fire) and its dependence on the driving force. We show that "energy", when injected uniformly (trees grow uniformly), is dissipated (trees burn) on a fractal. Scaling is checked by means of a Monte Carlo renormalization-group (MCRG) calculation. Our intention is to demonstrate in a toy model the emergence of scaling and fractal energy dissipation. Besides, at the critical points, "universality" may apply in analogy with equilibrium critical phenomena, and our study of this simple model may help to provide a consistent and theoretically sustainable phenomenological picture of a class of turbulent phenomena.

Our lattice model is defined in any dimensions, with the following simple rules:

(i) trees grow with a small probability p from empty sites at each time step;

(ii) trees on fire will burn down at the next time step;

(iii) the fire on a site will spread to trees at its nearest neighbor sites at the next time step.

There is only one parameter in the model, namely the growth rate of the trees [#1]. We now briefly sum-

[#1] Due to the discrete nature of our model the slow uniform growth (driving) is represented by small growth probability; the actual growth at each instance is random spatially. However, this randomness in driving does not affect the long-time and space behaviors of this mode. A continuously driven deterministic version of this model with uniform growth rate has been studied [4], which appears to exhibit the same scaling behavior. This indicates that the random noise in the our model is not important.

marize the main features of the model: The forest is characterized by a correlation length $\xi(p) \propto p^{-\nu}$; the critical point is at $p=0$; thus the system is critical as long as trees grow slowly. When the correlation length is larger than the size of the system, the fire will die out within a time of order the linear size of the system; when the correlation length is smaller than the size of the system, the forest fire is sustained. The spatial distribution of the fire (i.e. the energy dissipation) is on a fractal for length scales smaller than ξ; this is a manifestation of a critical point.

We simulate the forest fire on lattices of sizes up to 256×256 in 2d and 64×64×64 in 3d. Periodic boundary conditions are chosen. Starting from a homogeneous distribution of trees and fires, the forest fire evolves to a stationary state [#2] for $p>p_c$ ($p_c \propto L^{-1/\nu}$). The spatial distribution of the fires and trees (dissipation and energy storage) are the objects of our investigation. Fig. 1 shows a snapshot of the forest on fire, taken in the stationary state after an initial transient period. Note the coherent domains of trees separated by a fractal distribution of fires, indicating that the system is operating near a critical point. By measuring the number distribution $D(r)$ of fire at a distance r from a chosen site on fire ($D(r) \propto r^{D-1}$), we obtain the fractal dimension: $D=1.0\pm0.2$ in 2d and $D=2.5\pm0.2$ in 3d (fig. 2). This value of D agrees with experimental observations for turbulence [1]. Of course, this could be accidental. If one prefers the language from traditional equilibrium critical phenomena, the fire–fire corre-

[#2] Because of the finite size of the system the fire can die out accidentally in long simulations, even when the correlation length is smaller than the size of the system. A few fires, which are statistically insignificant, are added when the fire does die out in long simulations.

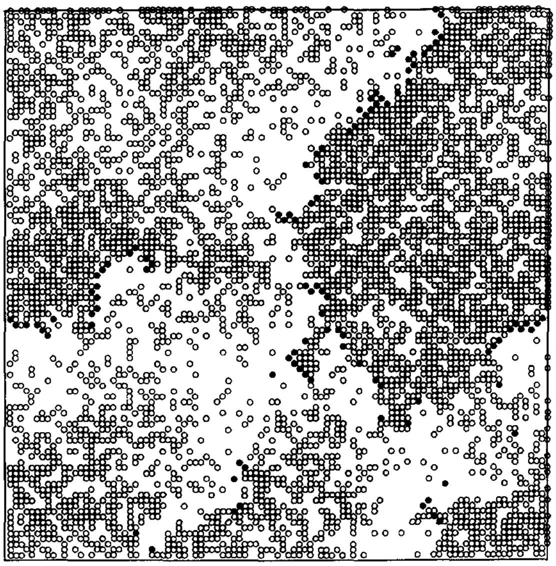

Fig. 1. Snapshot of forest fire in a 100×100 system. (○) Live trees, (●) burning trees.

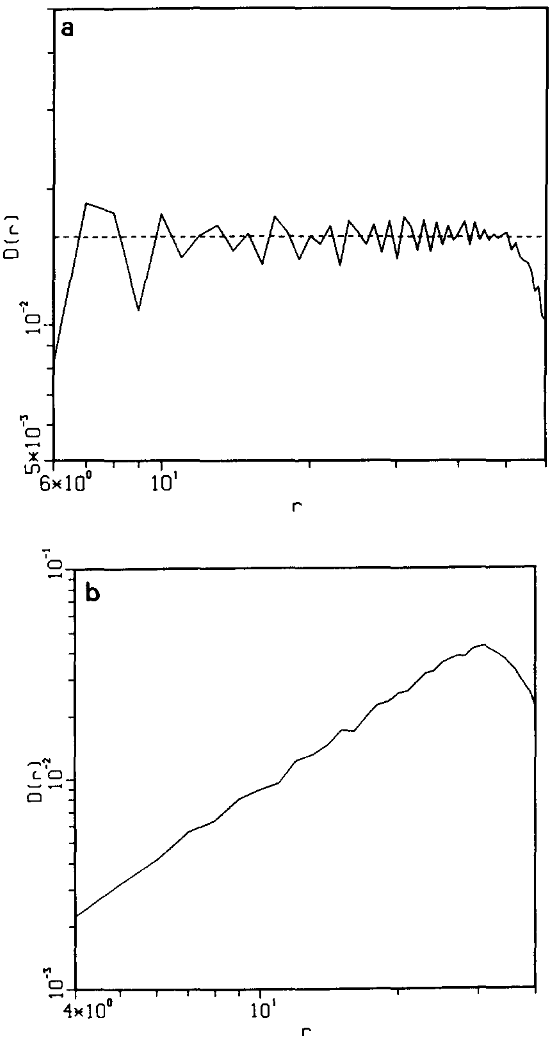

Fig. 2. Number distribution of fires at a distance r from a given fire site near the center of (a) a 120×120 system with $p=0.015$; (b) a 60×60×60 system with $p=0.0095$. Averages were taken over twenty thousand time steps.

lation function $G(x)=\langle f(x')f(x'+x)\rangle$ decays as $G(x)\propto|x|^{2-d-\eta}$ with $\psi\approx 1.0$ in 2d and $\eta\approx -0.5$ in 3d.

Given the fractal dimension, a simple energy conservation argument leads to a power-law dependence of ξ on p: The total number of trees burned down is equal to the total number of trees grown, which is $L^d p$, and is also equal to the total number of uncorrelated domains $(L/\xi)^d$ times the number of trees burned down in a given domain ξ^D. Thus, we have $\xi(p)\propto p^{-\nu}$, with $\nu=1/(d-D)$ ($\nu\approx 1.0$ in 2d and $\nu\approx 2.0$ in 3d).

A independent check on the power law dependence of ξ on p has been done using a Monte Carlo RG method. In two dimensions the calculations were performed for 32×32 and 64×64 lattices. We divide the lattice into 2×2 square blocks, and monitor the fire distribution of the blocked lattice. As the procedure is repeated, a hierarchy of renormalized fire distributions is obtained. Specifically, we record the number of boxes which contain fires in each blocked lattice, and average over the entire simulation. The average numbers $n(L, m, p)$ (m is the number of blocking iteration) after blocking down to lattices of size 4×4 and 2×2 are listed in table 1. The exponent ν can be calculated as follows: We find growth rates p_1 and p_2 such that

$$n(L=64, m+1, p_1)=n(L=32, m, p_2)\ . \qquad (1)$$

Because all lengths in the lattice, including the correlation length, are reduced by a factor of 2 from those of the original lattice after one blocking iteration, we have $\xi(p_1)/\xi(p_2)=2=(p_2/p_1)^\nu$. This relation is used to determine ν: Given $p_1=0.030$, matching is obtained with $p_2=0.060\pm 0.003$ for $m=3$ and 4; thus we have $\nu=1.0\pm 0.1$, which agrees with the value determined from the fractal dimension D. Numerical RG calculations in three dimensions on $32\times 32\times 32$ and $16\times 16\times 16$ lattices are not as conclusive. The matching on $2\times 2\times 2$ blocked lattices indicates $\nu\approx 1.4$, which is also in rough agreement with the value determined from the fractal dimension D.

Table 1
Values of the average number of boxes containing fires $n(L, m, p)$ for blocked lattices. The simulations were performed for 5×10^5 time steps on 64×64 lattices and for 2.5×10^5 time steps on 32×32 lattices (errors are statistical only)

	m	
	3 (4×4)	4 (2×2)
$n(64, m+1, p_1=0.030)$	10.00 ± 0.08	3.74 ± 0.04
$n(32, m, p_2=0.053)$	9.06 ± 0.02	3.67 ± 0.02
$n(32, m, p_2=0.057)$	9.57 ± 0.02	3.73 ± 0.02
$n(32, m, p_2=0.060)$	9.95 ± 0.02	3.78 ± 0.02
$n(32, m, p_2=0.062)$	10.17 ± 0.03	3.83 ± 0.03

In real turbulence, the Reynolds number is a combination of the size of the system and the driving force: $R=LV/\nu$. Similarly, we can define, by combining L and p, a "Reynolds number" R for the forest: $R=Lp^\nu$, which uniquely determines the behavior of the forest fire up to an overall scale. The transition to the steady "turbulent" state of the forest fire occurs at $R=R_c$, where the critical "Reynolds number" is $R_c\approx 1.8$ in 2d and is $R_c\approx 0.6$ in 3d. Note that the transition described here is a finite size crossover effect; so it is in real turbulence: the critical driving force goes to zero for infinite L [#3]. The dependence of the energy dissipation on the "Reynolds number" in the forest fire is also a power law: $E_d=p\propto R^\beta$ with $\beta=1/\nu$, where the first equation expresses stationarity.

The model may be rather directly applied to spreading of diseases, propagation of chemical activity, such as real fire. We believe that the model is simple enough to allow for explicit theoretical analysis, for instance renormalization group theories based on expansions around the upper critical dimension. Our study on a specific dynamical model shows explicitly that certain principles are viable: (a) Driven non-equilibrium systems may operate near critical points, and cannot be identified as low-dimensional chaos [6]. (b) Homogeneously injected energy is dissipated on a fractal. These principles may serve as important guidelines in studying realistic models.

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[#3] Pomeau [5] pointed out that certain transitions to turbulence are similar to the transition in directed percolation: Criticality requires fine-tuning of a control parameter. Our picture is entirely different: Criticality applies to the fully developed turbulent state itself, and the "transition" at finite Reynolds number is a crossover.

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