

Section 1 Vector space

- 1.1. n -dimensional real space
- 1.2. Inner (scalar) product and norm of vectors.
- 1.3. Linear mapping.
- 1.4. Linear independence of vectors.

1.1. n –dimensional real space

The set $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R} = \{x = (x_1, x_2, \dots, x_n) : x_i \in \mathbb{R} \text{ for } i \in \{1, \dots, n\}, n \in \mathbb{N}\}$ is **n –dimensional real space** (a set of n – elementary sequences).

An element $x = (x_1, x_2, \dots, x_n)$ in $\mathbb{R}^n$ is an n –dimensional **vector** (it has n coordinates); the real number x_i is said to be the **i th coordinate** of x .

Remark. 1. A row vector: $x = (x_1, x_2, \dots, x_n)$. A column vector: $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$.

2. The operation of converting a column vector into a row vector, and vice versa, is known as the

$$\text{transpose: } \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}^T = (x_1, x_2, \dots, x_n), \quad (x_1, x_2, \dots, x_n)^T = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

The set $\mathbb{R}^n$ is equipped with two operators, called **addition** of vectors and **scalar multiplication**.

Definition 1.1. (operations on vectors) Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, $a \in \mathbb{R}$.

- 1) The operator $\oplus : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$x \oplus y = (x_1, x_2, \dots, x_n) \oplus (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \in \mathbb{R}^n$ is called **addition** of vectors

- 2) The operator $\odot : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$a \odot x = a \odot (x_1, x_2, \dots, x_n) = (ax_1, ax_2, \dots, ax_n)$ is called **multiplication by scalar**.

Remark. To simplify notation: $x \oplus y = x + y$, $a \odot x = a \cdot x$.

Example 1.1 Find (and make a graph if possible) $x + y, a \cdot x, x \circ y$ where:

- a) $x = (4, 1), y = (2, 5), a = 2, a = -1$
- b) $x = (-2, 3, 5, 0), y = (1, 0, -2, 3), a = 3$

Basic properties of operations on vectors:

- 1) $\forall x, y \in \mathbb{R}^n : x \oplus y = y \oplus x$ commutative law of addition
- 2) $\forall x, y, z \in \mathbb{R}^n : x \oplus (y \oplus z) = (x \oplus y) \oplus z$ associative law of addition
- 3) $\exists \mathbf{0} = (0, 0, \dots, 0) \in \mathbb{R}^n \quad \forall x \in \mathbb{R}^n : x \oplus \mathbf{0} = \mathbf{0} \oplus x = x$,
- 4) $\forall x \in \mathbb{R}^n \quad \forall a, b \in \mathbb{R} : a \odot (b \odot x) = (ab) \odot x$ associative law of multiplication by scalar
- 5) $\forall x \in \mathbb{R}^n : 1 \odot x = x$,
- 6) $\forall x, y \in \mathbb{R}^n \quad \forall a \in \mathbb{R} : a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y)$ distributive law for multiplication by scalar
- 7) $\forall x \in \mathbb{R}^n \quad \forall a, b \in \mathbb{R} : (a + b) \odot x = (a \odot x) \oplus (b \odot x)$ distributive law for multiplication by scalar

1.2 Inner (scalar) product and norm of a vector

An inner (scalar) product space is a vector space with an additional structure called an inner product. This additional structure associates each pair of vectors in the space with a scalar quantity known as the **inner product** (scalar product or “dot” product) of the vectors.

Inner products allow the rigorous introduction of intuitive geometrical notions such as the length of a vector or the angle between two vectors. They also provide the means of defining orthogonality between vectors (zero inner product). Inner product is a pairing that takes two vectors and produces a real number.

Definition 1.2 Let $\mathbb{R}^n$ be a vector space.

A mapping $\circ : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is called an **inner product** in $\mathbb{R}^n$ iff it satisfies the following three axioms:

- | | | |
|----|---|-----------------------|
| 1) | $\forall x \in \mathbb{R}^n : (x \circ x) \geq 0$ and $(x \circ x = 0 \Leftrightarrow x = \mathbf{0})$ | positive-definiteness |
| 2) | $\forall x, y \in \mathbb{R}^n : x \circ y = y \circ x$ | conjugate symmetry |
| 3) | $\forall x, y \in \mathbb{R}^n \forall a, b \in \mathbb{R} : [(a \cdot x) + (b \cdot y)] \circ z = [a \cdot (x \circ z)] + [b \cdot (y \circ z)]$ | linearity |

Remark. $x \circ y = (x_1, x_2, \dots, x_n) \circ (y_1, y_2, \dots, y_n) = x_1 y_1 + x_2 y_2 + \dots + x_n y_n = \sum_{i=1}^n x_i y_i$, $x \circ y$ is called the **dot product** (or **standard Euclidean scalar product**).

Notation: $x \circ y = \langle x, y \rangle = \langle x | y \rangle$

Example 1.2 Determine if $\circ$ is an inner product in a space V if :

- $V = \mathbb{R}$, $x \circ y = x + y$
- $V = \mathbb{R}$, $x \circ y = xy$
- $V = \mathbb{R}^2$, $x \circ y = x_1 y_1 + x_2 y_2$
- $V = \mathbb{R}^2$, $x \circ y = x_1 y_1 + x_2 y_2 + 1$

When we have an inner product defined on a vector space we can define a norm and a distance.

Definition 1.3 Let $\mathbb{R}^n$ be a vector space with inner product $\circ$.

- The **norm** (or **length**) of a vector $x \in \mathbb{R}^n$ is a mapping $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by a formula:

$$\|x\| = \sqrt{x \circ x}.$$
- The **distance** between vectors $x, y \in \mathbb{R}^n$ is a mapping $d : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ defined by a formula:

$$d(x, y) = \|x - y\|.$$

Remarks.

- $\|x\| \geq 0$, $\|x\| = 0 \Leftrightarrow x = \mathbf{0}$
- $\|ax\| = |a| \cdot \|x\|$
- $\|-x\| = \|x\|$
- If $\|x\| \neq 0$, then the vector $v = \frac{x}{\|x\|}$ is a unit vector ($\|v\| = 1$) parallel to x

Example 1.3

- $V = \mathbb{R}^2$, $x \circ y = (x_1, x_2) \circ (y_1, y_2) = x_1y_1 + x_2y_2$,

$$\|x\| = \sqrt{x \circ x} = \sqrt{(x_1)^2 + (x_2)^2}$$

$$d(x, y) = \|x - y\| = \sqrt{(x - y) \circ (x - y)} = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

- $V = \mathbb{R}^n$, $x \circ y = (x_1, x_2, \dots, x_n) \circ (y_1, y_2, \dots, y_n) = x_1y_1 + x_2y_2 + \dots + x_ny_n$

$$\|x\| = \sqrt{x \circ x} = \sqrt{(x_1)^2 + (x_2)^2 + \dots + (x_n)^2}$$

$$d(x, y) = \|x - y\| = \sqrt{(x - y) \circ (x - y)} = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2}$$

Euclidean metric

Example 1.4 Compute $x \circ y$, $\|x\|$, $\|y\|$ and $\|x - y\|$ for vectors $x = (-1, 2)$, $y = (3, -1)$ and given inner product $x \circ y = x_1y_1 + x_2y_2$.

Definition 1.4 Let $\mathbb{R}^n$ be a vector space with inner product $\circ$.

- 1) Two vectors $x, y \in \mathbb{R}^n$ are called **orthogonal** (denoted by $x \perp y$) iff their inner product is zero:

$$x \perp y \Leftrightarrow x \circ y = 0.$$

- 2) Two vectors $x, y \in \mathbb{R}^n$ are called **orthonormal** iff x and y are orthogonal and both are unit vectors:

$$x \circ y = 0, \|x\| = \|y\| = 1.$$

Remark. Orthogonality is the mathematical formalization of the geometrical property of perpendicularity, and is a remarkably powerful tool that appears throughout the manifold applications of linear algebra. Two vectors are said to be orthogonal if their inner product vanishes.

Example 1.5 Decide whether the following vectors are orthogonal or orthonormal in the inner product

$$x \circ y = x_1y_1 + x_2y_2:$$

- a) $x = (-1, 2)$, $y = (6, 3)$,
- b) $x = (-1, 0)$, $y = (0, 1)$.

Remarks.

- Orthogonal vectors are also called **perpendicular**.
- The inner product of two nonzero vectors x and y can be define also as a number:

$$x \circ y = \|x\| \cdot \|y\| \cdot \cos\varphi \quad \text{where } \varphi \text{ is the angle between } x \text{ and } y.$$

- If two vectors are **parallel**, the angle between them is 0 (if they have the same direction) or π (if they have opposite directions), then: $x \parallel y \Leftrightarrow \left| \frac{x \circ y}{\|x\| \cdot \|y\|} \right| = 1$.

1.3. Linear mapping

Linear mapping is defined as a function between two vector spaces that preserves vector addition and scalar multiplication.

Definition 1.5. Let $\mathbb{R}^n, \mathbb{R}^m$ be two vector spaces. A function $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a **linear mapping** (linear transformation) iff $\forall x, y \in \mathbb{R}^n \quad \forall a \in \mathbb{R}$:

- 1) $L(x + y) = L(x) + L(y)$ additivity
- 2) $L(a \cdot x) = a \cdot L(x)$ homogeneity

Example 1.6 Let a function $f: \mathbb{R} \rightarrow \mathbb{R}$ be given.

- 1) $f(x) = ax$,
- 2) $f(x) = ax + b$, where $a, b \in \mathbb{R}$ and $b \neq 0$
- 3) $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$

CAUTION: linear function $\neq$ linear mapping.

Theorem 1.1. (necessary condition for linear mapping)

If a function $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear mapping then $L(\mathbf{0}) = \mathbf{0}$.

Example 1.7 Let a function be given.

- 1) $L: \mathbb{R} \rightarrow \mathbb{R} \quad L(x) = ax + b$, where $a, b \in \mathbb{R}$ and $b \neq 0$
- 2) $L: \mathbb{R}^2 \rightarrow \mathbb{R} \quad L(x, y) = x^2 + y$
- 3) $L(x_1, x_2) = (x_1 + x_2, 2x_1 + 1)$

Remark. Every linear mapping $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ can be represented as a rectangular array of numbers called a **matrix representation** M_L (or a **matrix**) of the mapping L :

- This mapping has a general form: $L(x_1, x_2, \dots, x_n) = (a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n, a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n, \dots, a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n)$.

- The matrix representation of the mapping L is: $M_L = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$.

Example 1.8

- 1) Let be given a linear mapping $L: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that: $L(x_1, x_2) = (3x_1 - 2x_2, x_1 + 4x_2, 5x_1)$.

The matrix representation of the mapping L (in canonical bases) is: $M_L = \begin{bmatrix} 3 & -2 \\ 1 & 4 \\ 5 & 0 \end{bmatrix}$.

- 2) Let be given a matrix $M_L = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 0 & -2 \end{bmatrix}$. This matrix is the matrix representation of the mapping

$L: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ such that: $L(x_1, x_2, x_3) = (x_1 - 3x_2 + x_3, 4x_1 - 2x_3)$.

1.4. Linear independence of vectors

Let be given vectors $v_1, v_2, \dots, v_k \in \mathbb{R}^n$ and scalars $a_1, a_2, \dots, a_k \in \mathbb{R}$. A sum of the form:

$a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_k \cdot v_k = \sum_{i=1}^k a_i \cdot v_i$ is known as a **linear combination of vectors** $v_1, v_2, \dots, v_k$.

Definition 1.6

1. The vectors $v_1, v_2, \dots, v_k \in \mathbb{R}^n$ are called **linearly dependent** if there exist scalars $a_1, a_2, \dots, a_k \in \mathbb{R}$, not all equal zero, such that $a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_k \cdot v_k = \mathbf{0}$.
2. The vectors $v_1, v_2, \dots, v_k \in \mathbb{R}^n$ are called **linearly independent** if $a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_k \cdot v_k = \mathbf{0}$ only for $a_1 = a_2 = \dots = a_k = 0$.

Example 1.9

1. The vectors $v_1 = (1, 2, -1), v_2 = (0, 3, 1), v_3 = (-1, 4, 3)$
2. The vectors $v_1 = (1, 2, -1), v_2 = (0, 3, 1)$

Example 1.10 Decide whether or not the following vectors are linearly independent in the space $\mathbb{R}^3$:

- a) $x_1 = (1, 0, 1), x_2 = (1, 1, 1), x_3 = (0, -1, 0)$
- b) $x_1 = (1, 0, 2), x_2 = (1, 1, 0), x_3 = (2, -1, 0)$

Remark.

- 1) In $\mathbb{R}^2$, the vectors $e_1 = (1, 0), e_2 = (0, 1)$ are linearly independent and for any $(x, y) \in \mathbb{R}^2$:
 $(x, y) = x \cdot (1, 0) + y \cdot (0, 1) = x \cdot e_1 + y \cdot e_2$.
- 2) In $\mathbb{R}^n$, $e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_n = (0, 0, \dots, 0, 1)$ are linearly independent and for any $x \in \mathbb{R}^n$: $x = (x_1, x_2, \dots, x_n) = x_1 \cdot (1, 0, \dots, 0) + x_2 \cdot (0, 1, \dots, 0) + \dots + x_n \cdot (0, 0, \dots, 0, 1) = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$.
- 3) The vector e_i is called i th **canonical unit vector**.

Remark.

1. A set of m vectors in $\mathbb{R}^n$ must be linearly dependent if $m > n$.
2. If vectors $v_1, v_2, \dots, v_k \in \mathbb{R}^n$ are linearly independent then $v_1, v_2, \dots, v_{k-1}$ are linearly independent.

Dictionary

associative	łączny
commutative	przemienny
distance (n.)	odległość
distributive	rozdzielny
Euclidean space (inner product space)	przestrzeń euklidesowa (przestrzeń wektorowa z iloczynem skalarnym)
inner product (dot product)	iloczyn skalarny
length (n.)	długość
linear combination	kombinacja liniowa
linear space (vector space)	przestrzeń liniowa (wektorowa)
linearly dependent	liniowo zależny
linearly independent	liniowo niezależny
norm (n.)	norma
orthogonal vectors	wektory ortogonalne (prostopadłe)
orthonormal vectors	wektory ortonormalne (prostopadłe i o długości 1)
scalar multiplication	mnożenie wektora przez skalar
unit vector	wektor jednostkowy
vector space (linear space)	przestrzeń wektorowa (liniowa)