Complex numbers were defined in XVI century and became an answer for a question about roots of algebraic equations. It is well known that even quadratic equations, for instance, $x^2 + 1 = 0$, have no solution in $\mathbb{R}$ – in our example a quantity $\sqrt{-1}$ makes no sense in $\mathbb{R}$. Therefore the cumberstone of the theory of complex numbers was to give meaning to the roots of negative numbers.

The theory was formulated first by Italian mathematician R. Bombelli in 1572 (paper *L'Algebra*). In 1748 L. Euler, a Swiss mathematician introduced this idea to analytical calculus. German C.F. Gauss and Irish W.R. Hamilton developed a strict theory of complex numbers independently in the XIX century. Nowadays complex numbers are a common tool in mathematics, physics and technical sciences.

1.1 Construction of set of complex numbers

Definition 1.1. Let $i \coloneqq \sqrt{-1}$ and $\mathbb{C} = \{z = a + bi \ : \ a \in \mathbb{R} \ \wedge \ b \in \mathbb{R}\}$.

Elements of the set $\mathbb{C}$ are called **complex numbers**.

a is a **real part** of complex number $z = a + bi$, i.e. $\mathrm{Re}\,(a + bi) = a$.

b is an **imaginary part** of complex number $z = a + bi$, i.e. $\mathrm{Im}\,(a + bi) = b$.

Complex number $i = \sqrt{-1}$ is **imaginary unit**.

Remark. If for a complex number z $\mathrm{Im}\, z = 0$, then $z \in \mathbb{R}$. Therefore the set of complex numbers is a superset of $\mathbb{R}$ and any real number is also a complex number.

Definition 1.2. (operations on complex numbers)

Let $z_1 = a_1 + ib_1,\ z_2 = a_2 + ib_2,\ z_1, z_2 \in \mathbb{C}$.

Addition of complex numbers: $z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$

Multiplication of complex numbers:

$$z_1 \cdot z_2 = (a_1 + ib_1) \cdot (a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

Example 1.1 Given $z_1 = 1 + i, z_2 = 2 - i$ find: $z_1 + z_2,\ -z_2,\ z_1 - z_2, z_1 \cdot z_2,\ z_2^{-1},\ \frac{z_1}{z_2}$.

Remarks.

1. If $z_1, z_2 \in \mathbb{R}$, i.e. $\operatorname{Im} z_1 = \operatorname{Im} z_2 = 0$, then the above definitions lead to the standard operations of addition and multiplication in $\mathbb{R}$.
2. Set of complex numbers $\mathbb{C}$ may be identified with the Cartesian plane $\mathbb{R}^2$.

Theorem 1.1. (properties of operations on complex numbers)
Let $z, z_1, z_2 \in \mathbb{C}$. Then:

1. $z_1 + z_2 = z_2 + z_1, \quad z_1 \cdot z_2 = z_2 \cdot z_1$ addition and multiplication are both commutative
2. $(z_1 + z_2) + z_3 = z_2 + (z_1 + z_3), \; (z_1 \cdot z_2) \cdot z_3 = z_2 \cdot (z_1 \cdot z_3)$ addition and multiplication are associative;
3. $0 \equiv 0 + i \cdot 0$ is the neutral element of addition;
4. $-z = -a + i(-b)$ is the opposite element to z (inverse element in sense of addition);
5. $1 \equiv 1 + i \cdot 0$ is the neutral element of multiplication;
6. any nonzero complex number $z \neq 0$ has its inverse $\frac{1}{z} = z^{-1} = \frac{a}{a^2+b^2} + i \cdot \frac{-b}{a^2+b^2}$ (inverse element in sense of multiplication);
7. $(z_1 + z_2) \cdot z_3 = z_1 \cdot z_3 + z_2 \cdot z_3$ multiplication is distributive with respect to addition;
8. $z \cdot 0 = 0 \cdot z = 0$
9. $z_1 \cdot z_2 = 0 \;\Rightarrow\; z_1 = 0 \vee z_2 = 0$
10. $(-z_1) \cdot z_2 = -(z_1 \cdot z_2)$
11. $-1 \cdot z = -z$

Basing on the definitions of opposite and inverse elements, we define substraction and division in $\mathbb{C}$ in the following way:

$$z_1 - z_2 := z_1 + (-z_2); \qquad \frac{z_1}{z_2} := z_1 \cdot z_2^{-1}$$

Definition 1.3. (complex conjugate) Number $\boldsymbol{\bar{z} = a - ib}$ is called **complex conjugate** of $z = a + ib$.

Theorem 1.2. (properties of complex conjugate)
Let $z, z_1, z_2 \in \mathbb{C}$. Then:

a) $\overline{(\bar{z})} = z$
b) $z + \bar{z} = 2 \operatorname{Re} z, \quad z - \bar{z} = 2 \operatorname{Im} z$
c) $\bar{z} = z \Leftrightarrow \operatorname{Im} z = 0$
d) $\bar{z} = 0 \Leftrightarrow z = 0$
e) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
f) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$
g) $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$
h) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ (if $z_2 \neq 0$)

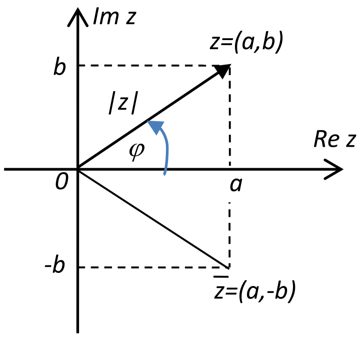
Example 1.2. Let $z = 2 + 3i$. Find $\bar{z}$ and calculate $\bar{z}z$, $z + \bar{z}$, $z - \bar{z}$.

1.2 Algebraic, trigonometric and exponential forms of complex number

A complex number as $z = a + ib$ is called **algebraic form** of z.

Complex number $z = a + ib$ is represented by the point (a, b) on the plane or by a pair $(|z|, \varphi)$, which denotes the distance of z from the origin and the angle between positive half-real axis and chord Oz.

Horizontal axis is called **real axis**, and vertical – **imaginary** one.



Remarks.

1. Set of complex numbers $\mathbb{C}$ may be identified with a real vector space $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ of dimension two called a **Cartesian (or complex) plane** with operations:
   - $(a, b) + (c, d) = (a + c, b + d)$
   - $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$
2. In a **rectangular coordinate system**, we were plotting points based on an ordered pair of (x, y).
3. In a **polar coordinate system** each point on a plane is determined by a distance from a reference point and an angle from a reference direction.

Definition 1.4. Let $z \in \mathbb{C}$, $z = a + ib$. Then:

a) $|z| = \sqrt{a^2 + b^2}$ is called the **modulus** of z.

b) an angle φ, such that $\boldsymbol{\cos\varphi = \frac{a}{|z|}}$ and $\boldsymbol{\sin\varphi = \frac{b}{|z|}}$, is called the **argument** of z : $\arg(z) = \varphi$.

c) an angle $\varphi \in [0, 2\pi)$, such that $\cos\varphi = \frac{a}{|z|}$ and $\sin\varphi = \frac{b}{|z|}$, is called the **principal argument** of z: $\mathrm{Arg}(z) = \varphi$.

This definition leads us to **trigonometric form** of complex number:

$$z = a + ib = |z|\left(\frac{a}{|z|} + i \cdot \frac{b}{|z|}\right) = |z|(\cos\varphi + i\sin\varphi)$$

Trigonometric form of a complex number: $z = |z|(\cos\varphi + i\sin\varphi)$

Trigonometric form of a complex number is extremely useful in finding powers and roots of it.

Theorem 1.3. (operations on trigonometric forms)

Let $z_1, z_2 \in \mathbb{C}$, $z_1 = |z_1|(\cos\varphi_1 + i\sin\varphi_1), z_2 = |z_2|(\cos\varphi_2 + i\sin\varphi_2)$. Then:

a) $z_1 \cdot z_2 = |z_1| \cdot |z_2|(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2))$

b) $\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2))$ for $z_2 \neq 0$

Example 1.3. a) Let $z_1 = 1 - i\sqrt{3}$, $z_2 = 1 + i$. Find their trigonometric forms and numbers $z_1 \cdot z_2$, $\frac{z_1}{z_2}$.

Exponential form of complex number: $z = |z|e^{i\varphi}$

Theorem 1.4. (operations on trigonometric forms)

Let $z_1, z_2 \in \mathbb{C}$, $z_1 = |z_1|e^{i\varphi_1}, z_2 = |z_2|e^{i\varphi_2}$. Then:

a) $z_1 \cdot z_2 = |z_1| \cdot |z_2|e^{i(\varphi_1+\varphi_2)}$; b) $\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}e^{i(\varphi_1-\varphi_2)}$ for $z_2 \neq 0$.

Example 1.3. b) Find exponential forms for numbers z_1, z_2, $z_1 \cdot z_2$, $\frac{z_1}{z_2}$, where $z_1 = 1 - i\sqrt{3}$, $z_2 = 1 + i$.

1.3 Power and root of complex number

Definition 1.5. Power of a complex number z of natural exponent is defined by recurrence:

$$z^1 = z, \qquad z^{k+1} = z \cdot z^k \quad \text{for } k \in \mathbb{N}$$

It is possible to extend this definition for *(negative) integer exponent*, if $z \neq 0$:

$$z^{-k} = (z^{-1})^k \quad \text{for} \quad k \in \mathbb{N}$$

The following theorem provides a tool to find quickly any power (especially large) of a complex number.

Theorem 1.5. (de Moivre's formula) Let $n \in \mathbb{N}$, $z \in \mathbb{C}$, $z = |z|(\cos\varphi + i\sin\varphi)$.

Then: $z^n = |z|^n(\cos n\varphi + i\sin n\varphi)$ and $z^{-n} = |z|^{-n}(\cos(-n\varphi) + i\sin(-n\varphi))$.

Example 1.4. Find $\left(1 - i\sqrt{3}\right)^{24}$.

The root of a complex number is defined in analogical way as in set of reals.

Definition 1.6. Let $n \in \mathbb{N}$, $z \in \mathbb{C}$. Number $x \in \mathbb{C}$ is a **root of a complex number** z if $x^n = z$.

Remark. Each complex number z has precisely n distinct roots of nth degree.

Theorem 1.6. (de Moivre's formula for roots) Let $n \in \mathbb{N}$, $z \in \mathbb{C}$, $z = |z|(\cos\varphi + i\sin\varphi)$.
Then there exist n solutions to the equation $x^n = z$, which are given by

$$x_k = \sqrt[n]{|z|}\left(\cos\frac{\varphi + 2k\pi}{n} + i\sin\frac{\varphi + 2k\pi}{n}\right), \quad k \in \{0,1,\dots,n-1\}.$$

Remark. For any $k \in \{0,1,\dots,n-1\}$ $|x_k| = \sqrt[n]{|z|}$ and arguments of successive roots increase by $\frac{2\pi}{n}$, therefore all roots x_k are equally spaced around the circle of radius $|x_k|$.

Example 1.5. Find $\sqrt[3]{-1}$.

Example 1.6. Solve the quadratic equation: $x^2 = z$.

1.4 Solution to polynomial equation in $\mathbb{C}$

Theorem 1.7. (Fundamental theorem of algebra)

Let $P \in \mathbb{C}[x]$, i.e. $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $\forall i\colon\ a_i \in \mathbb{C}$.

Then there exist n roots of $P(x) = 0$.

Theorem 1.8. Let $P \in \mathbb{R}[x]$ and $P(z) = 0$. Then $P(\bar{z}) = 0$.

Example 1.7. Solve the equation $x^3 + 4x^2 + 5x = 0$.

Example 1.8. Solve the equation: $x^2 + (1 + 4i)x - (5 + i) = 0$.