Matrix theory is a powerful mathematical tool for dealing with data as a whole rather than individual items. In both pure and applied mathematics, many situations deal with rectangular arrays of numbers. In fact, in many branches of business and biological and social sciences, it is necessary to express and use a set of numbers in a rectangular array. This array we called a *matrix*. Matrices can be added, subtracted and multiplied. They also possess many of the algebraic properties of numbers. Matrix algebra is the study of these properties.

## 1.1 Matrix – definition and specific matrices

**Definition 1.1 (formal)** **A matrix** $A$ of $m$ rows and $n$ columns is a mapping defined on a Cartesian product of two sets into a nonempty set $V \neq \emptyset$: $A:\ \{1,2,\dots,m\} \times \{1,2,\dots,n\} \to V$

$$A:\ (i,j) \to a_{ij} \in V$$

For $V = \mathbb{R}$ we have a *real matrix*.

**Remarks.**

- A matrix (plural: matrices) can be informally defined as a rectangular array of elements that are enclosed by a pair of brackets:

$$A = A_{(m,n)} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \begin{matrix} \leftarrow 1st\ row \\ \leftarrow 2nd\ row \\ \vdots \\ \leftarrow ith\ row \\ \vdots \\ \leftarrow mth\ row \end{matrix}$$

$$\begin{matrix} \uparrow & \uparrow & & \uparrow & & \uparrow \\ 1st & 2nd & & jth & & nth\ column \end{matrix}$$

- a ***row of a matrix*** is a horizontal vector in matrix $A$; $[a_{i1}, a_{i2}, \dots, a_{in}] = r_i$ is an $i$th row,
- a ***column of a matrix*** is a vertical vector in matrix $A$; $[a_{1j}, a_{2j}, \dots, a_{mj}] = c_j$ is a $j$th column,
- the numbers $a_{ij}$ in matrix $A$ are called the ***elements*** or ***entries*** or ***components*** of the matrix,
- each element $a_{ij}$ of a matrix $A$ has two indices: the row index $i$, and the column index $j$ ($a_{ij}$ is an entry in the $i$th row and $j$th column),
- a matrix $A$ with $m$ rows and $n$ columns contains $m \cdot n$ elements,
- the number of rows and columns, $m \times n$ (read as: "$m$ by $n$"), is the ***dimension*** or ***size*** of the matrix; in general, $m \times n$ matrix has $m$ rows and $n$ columns.

**Example 1.1.**

## Specific matrices

**1.** A ***column matrix*** or *column vector* of height $m$ ($m \times 1$ matrix, $n = 1$) is a matrix with 1 column of elements:

$$A = \begin{bmatrix} \\ \\ \\ \end{bmatrix}_{3\times 1}$$

A ***row matrix*** or *row vector* of length $n$ ($1 \times n$ matrix, $m = 1$) is a matrix with 1 row of elements:

$$A = [\qquad\qquad\qquad\qquad]_{1\times 4}$$

**2.** A ***zero matrix*** is a matrix in which all entries are zero. We use the symbol **0** to represent a zero matrix of any dimensions: $\mathbf{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$

**3.** A ***square matrix*** has as many rows as columns, i.e. $m = n$: $A = A_{(n,n)} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}_{n\times n}$

In the square matrix $A = \left[a_{ij}\right]_{n\times n}$ the entries for which $i = j$, namely $\{a_{11}, a_{22}, \dots, a_{nn}\}$, $n \in \mathbb{N}$, form a ***principal diagonal*** (or ***main diagonal*** or ***primary diagonal*** or ***diagonal entries***) of $A$.

Elements $\left\{a_{1n}, a_{2(n-1)}, \dots, a_{n1}\right\}$ form a ***secondary diagonal***.

**4.** A ***diagonal matrix*** is a square matrix with all non-diagonal elements equal zero:

$\forall i,j \in \{1,2,\dots,n\}; i \neq j:\ a_{ij} = 0 \qquad D = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}_{n\times n}$ e.g. $D = \begin{bmatrix} \dots & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & \dots \end{bmatrix}_{3\times 3}$

**5.** The ***identity (unit) matrix*** is a diagonal matrix such that every diagonal element is equal to 1:

$a_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases} \qquad I = I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}_{n\times n}$ e.g. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

**Notation:**

- $M_{m\times n}(\mathbb{R})$ is a set of all real matrices of dimensions $m \times n$.
- $M(m,n)$ is a set of all matrices of dimensions $m \times n$.
- $M_n(\mathbb{R})$ is a set of all real square matrices of dimensions $n \times n$.

## 1.2 Operations on (with) matrices

- **Multiplication by scalar (scalar multiplication)**

Let $A = [a_{ij}]$ be an $m \times n$ matrix and $k \in \mathbb{R}$ be a real number, called a *scalar*. The product of the matrix $A$ by the scalar $k$, called **scalar multiplication**, is the $m \times n$ matrix: $k \cdot A = [k \cdot a_{ij}]_{m\times n}$.

**Example 1.2** Multiply given matrix $A$ by scalar $k$.

- **Transposition**

The *transpose of a matrix A*, denoted by $A^T$, is found by interchanging rows and columns of $A$.

Let $A = [a_{ij}]$ be an $m \times n$ matrix. The **transpose of the matrix** $A$ is the $n \times m$ matrix: $A^T = [a_{ji}]_{n\times m}$

**Example 1.3** Find the transpose of $A$.

**Remark** Let $A \in M_{m\times n}(\mathbb{R})$, $B \in M_{n\times p}(\mathbb{R})$, $I \in M_n(\mathbb{R})$ and $k \in \mathbb{R}$. Then:

**1.** $I^T = I$ **2.** $(A^T)^T = A$ **3.** $(k \cdot A)^T = k \cdot A^T$ **4.** $(A \cdot B)^T = B^T \cdot A^T$

- **Matrix addition and subtraction (sum of two matrices)**

If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are two matrices of the same dimensions, then the **sum and the difference** are defined as the $m \times n$ matrices: $A + B = [a_{ij} + b_{ij}]_{m\times n}$, $A - B = [a_{ij} - b_{ij}]_{m\times n}$

**Caution !** The sum or difference of two matrices **DOES NOT EXIST** if dimensions of matrices are not the same.

**Example 1.4** Add and subtract given matrices.

**Remark** Let $A, B, C, \mathbf{0} \in M_{m\times n}(\mathbb{R})$ and $k, l \in \mathbb{R}$. Then:

1. $A + B = B + A$ commutativity of addition
2. $A + (B + C) = (A + B) + C$ associativity of addition
3. $A + \mathbf{0} = A$ the zero matrix is an additive identity
4. $k \cdot (A + B) = (k \cdot A) + (k \cdot B)$ scalar multiplication is distributive over matrix addition
5. $(k + l) \cdot A = (k \cdot A) + (l \cdot A)$ scalar multiplication is distributive over the addition of numbers
6. $(k \cdot l) \cdot A = k \cdot (l \cdot A)$

- **Matrix multiplication (product of two matrices)**

**Caution !** We **DO NOT** multiply two matrices by multiplying their corresponding members.

**A product of row matrix and column matrix:**

If $A = [a_{11} \quad a_{12} \quad \cdots \quad a_{1m}]$ is a row matrix $(1 \times m)$ and $B = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix}$ is a column matrix $(m \times 1)$, then:

$A \cdot B = [c_{11}]$ to be $1 \times 1$ matrix (a number), where

$$c_{11} = a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} + \cdots + a_{1m} \cdot b_{m1}.$$

$B \cdot A = [c_{ij}]$ to be $m \times m$ matrix, where $c_{ij} = b_{i1} \cdot a_{1j}$.

**The Falck's diagrams:**

$$\begin{array}{c|c} B = & \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix}_{m \times 1} \\ \hline A = [a_{11} \quad a_{12} \quad \cdots \quad a_{1m}]_{1 \times m} & [c_{11}]_{1 \times 1} = A \cdot B \end{array}$$

$$\begin{array}{c|c} A = & [a_{11} \quad \cdots \quad a_{1j} \quad \cdots \quad a_{1m}]_{1 \times m} \\ \hline B = \begin{bmatrix} b_{11} \\ \vdots \\ b_{i1} \\ \vdots \\ b_{m1} \end{bmatrix}_{m \times 1} & \begin{bmatrix} c_{11} & \cdots & \cdots & \cdots & c_{1m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \cdots & \cdots & c_{ij} & \cdots & \cdots \\ \vdots & & \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mm} \end{bmatrix}_{m \times m} = B \cdot A \end{array}$$

**Example 1.5 a)** For given matrices $A$ and $B$ compute $A \cdot B$ and $B \cdot A$.

**A product of two matrices:**

If $A = [a_{ij}]_{m\times p}$ is an $m \times p$ matrix and $B = [b_{ij}]_{p\times n}$ is a $p \times n$ matrix, then their product

$$A \cdot B = [c_{ij}]_{m\times n} \quad \text{is an } m \times n \text{ matrix with elements } c_{ij}\text{, such that:}$$

$$c_{ij} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + a_{i3} \cdot b_{3j} + \cdots + a_{ip} \cdot b_{pj}$$

($c_{ij}$ is an inner product of $i$th row of the first matrix and $j$th column of the second matrix: $c_{ij} = r_i^A \circ c_j^B$).

**The Falck's diagram:**

$$\begin{array}{r|l}
 & B = \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & & \vdots & \ddots & \vdots \\ b_{p1} & \cdots & b_{pj} & \cdots & b_{pn} \end{bmatrix}_{p\times n} \\ \hline
A = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ip} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix}_{m\times p} & \begin{bmatrix} c_{11} & \cdots & \cdots & \cdots & c_{1n} \\ \cdots & \cdots & c_{ij} & \cdots & \cdots \\ \vdots & & \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mn} \end{bmatrix}_{m\times n} = A \cdot B
\end{array}$$

**Example 1.5 b)** For given matrices $A$ and $B$ compute $A \cdot B$ and $B \cdot A$.

**Remark** Let $A \in M_{m\times n}(\mathbb{R})$, $B \in M_{n\times p}(\mathbb{R})$, $C \in M_{p\times r}(\mathbb{R})$, $I \in M_n(\mathbb{R})$. Then:

1. $A \cdot B \neq B \cdot A$
2. $A \cdot (B \cdot C) = (A \cdot B) \cdot C$ — associativity of multiplication
3. $I \cdot A = A \cdot I = A$ — identity matrix is a multiplication identity

**Remark** If $A \in M_n(\mathbb{R})$ is a square matrix, we can define *powers* of it: $A^2 = A \cdot A$, $A^3 = A^2 \cdot A$, and so on. In general for any positive integer $k$ we have $A^{k+1} = A^k \cdot A$, where we take $A^0 = I_n$.

**Example 1.6** For a matrix $A$ compute $A^2$.

## 1.3 Determinant of square matrix

Let $A \in M_n(\mathbb{R})$. ***Determinant*** is a real number uniquely associated with square matrix.

We will denote the determinant of the matrix $A$ by $\det A$ or $|A|$ or $\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}_{n\times n}$

**Definition 1.2**
The **minor** $M_{ij}$ of dimension $n-1$ (**subdeterminant**) belonging to the element $a_{ij}$ of a square matrix $A$ of order $n \geq 2$ is the determinant of the matrix obtained by deleting the $i$th row and the $j$th column of $A$.

**Example 1.7** Given the determinant $det\begin{bmatrix} -2 & 3 & 0 \\ 5 & 1 & -2 \\ 7 & -4 & 8 \end{bmatrix}$ write the minor of each of the following elements: $a_{11}, a_{23}, a_{31}$.

**Definition 1.3** The **cofactor** $C_{ij}$ of an element $a_{ij}$ of a square matrix $A$ is given by

$$C_{ij} = (-1)^{i+j} \cdot M_{ij} \qquad \text{where } M_{ij} \text{ is the minor of } a_{ij}.$$

**Example 1.7 (cont.)** Given the determinant $det\begin{bmatrix} -2 & 3 & 0 \\ 5 & 1 & -2 \\ 7 & -4 & 8 \end{bmatrix}$ write the cofactor of elements: $a_{11}, a_{23}, a_{31}$.

**Remark** The matrix $A^C$ formed by all the cofactors of the elements in a matrix $A = [a_{ij}]_{n\times n}$ is called the ***cofactor matrix*** $A$: $A^C = [C_{ij}]_{n\times n}$.

**Definition 1.4 (by induction)**

**1.** $n = 1$ **(first-order determinant)**

A first-order determinant is defined as follows: $A = [a_{11}]$  $\det A = a_{11}$

**2.** $n > 1$ **($n$th-order determinant)**

The value of a determinant may be found by multiplying every element in any row (or column) by its cofactor and summing the products. This method is called *expanding by cofactors* (or *Laplace expansion rule*).

For the $r$th row of $A$, the determinant of $A$ is:

$$\det A = \det\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rj} & \cdots & a_{rn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = a_{r1} \cdot C_{r1} + a_{r2} \cdot C_{r2} + \cdots + a_{rn} \cdot C_{rn}\,.$$

For the $r$th column of $A$, the determinant of $A$ is:

$$\det A = \det\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2r} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} & \cdots & a_{mn} \end{bmatrix} = a_{1r} \cdot C_{1r} + a_{2r} \cdot C_{2r} + \cdots + a_{nr} \cdot C_{nr}\,.$$

**Remarks**

- A second-order determinant of a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is given as follows:

$$\det A = a_{11} \cdot C_{11} + a_{12} \cdot C_{12} = a_{11} \cdot (-1)^{1+1} \cdot a_{22} + a_{12} \cdot (-1)^{1+2} \cdot a_{21} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

- A third-order determinant $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is given as follows (the ***Sarrus rule***):

**Example 1.8** Evaluate the determinant of the given matrix $A$.

**Basic properties of the determinants of matrices:** Let $A, B, I \in M_n(\mathbb{R})$ and $k \in \mathbb{R}$. Then:

**1.** $\det I = 1$
**2.** $\det A^T = \det A$
**3.** $\det(kA) = k^n \det A$
**4.** $\det(A \cdot B) = \det A \cdot \det B$
**5.** If any of two rows (columns) of $A$ are interchanged, the determinant of the new matrix equals $-\det A$.
**6.** The determinant is zero if:
**a)** two rows (two columns) of $A$ are identical,
**b)** a matrix has a row (column) containing all zeros,
**c)** two rows (two columns) of a matrix are proportional.
**7.** If the result of multiplying a row (column) of $A$ by a constant is added to another row (column) of $A$, the determinant of the new matrix is equal to $\det A$.

## 1.4. Elementary row or column operations

**Elementary row or column operations (elementary transformations) are:**
1. The **interchanging** of any two rows (columns) of a matrix ($r_i \leftrightarrow r_j$; $\leftrightarrow$ means "change").
2. **Multiplying** of each element of a row (column) by the same nonzero number ($kr_i \mapsto r_i'$; $\mapsto$ means "replaces").
3. The **replacement** of any row (column) by the sum of that row (column) and a nonzero multiple of some other row (column) ($r_i + kr_j \mapsto r_i'$).

**Applications:**

1. Calculating determinants: before expanding by cofactors we may transform a determinant with the help of elementary transformations into a form such that it contains as many zeros as possible or into a triangular matrix.
2. Finding the inverse of a matrix.
3. Solving general system of linear equations (the Gauss or Gauss-Jordan elimination methods).

**Properties:**

- Operation 1) changes determinant of a matrix ($\det A' = -\det A$).
- Operation 2) changes determinant of a matrix ($\det A' = k \det A$).
- Operation 3) does NOT change determinant of a matrix ($\det A' = \det A$).

**Example 1.9** Evaluate the determinant of the given matrix $A$.

## 1.5 Inverse of a matrix

**Definition 1.5.** Let $A \in M_n(\mathbb{R})$. A matrix $A^{-1} \in M_n(\mathbb{R})$ is called an **inverse matrix** of $A$ with respect to multiplication (in short: **inverse of** $A$), iff: $A \cdot A^{-1} = A^{-1} \cdot A = I.$

**Remarks**

- A square matrix $A$ has at most one inverse.
- A square matrix $A$ which has the inverse matrix is called ***invertible***.

**Caution !** Not all square matrices are invertible.

**Remark** A square matrix $A$ is invertible iff $\det A \neq 0$.

- ***Inverse matrix using cofactors***

**Theorem 1.1.** Let $A \in M_n(\mathbb{R})$, $\det A \neq 0$ and $A^C \in M_n(\mathbb{R})$ be its cofactor matrix. Then $A$ is invertible and $A^{-1} = \frac{1}{detA} \cdot [A^C]^T.$

**Remark** $[A^C]^T$ is called the ***adjoint*** (or ***adjugate***) matrix.

Procedure for calculating the inverse $A^{-1}$ of a matrix $A$ using the cofactors:

a) check if $\det A \neq 0$,
b) find the cofactor of each element,
c) replace each element by its cofactor,
d) find the transpose of the matrix found in (c),
e) multiply the matrix in (d) by $\frac{1}{detA}$.
The result is $A^{-1}$.

**Example 1.10a** If possible find inverse of given matrices.

- ***Inverse matrix using elementary row (column) operations* (the *Gauss-Jordan reduction method*)**

Procedure for calculating the inverse $A^{-1}$ of a matrix $A$ using elementary row (column) operations:

**a)** first we form a new matrix consisting, on the left, of the matrix $A$, on the right, of the corresponding identity matrix $I$: $[A : I]$,

**b)** now we use elementary row operations to produce matrix: $[I : A^{-1}]$,

**c)** we attempt to transform $A$ to an identity matrix, but whatever operations we perform, we do on the identity matrix,

**d)** the matrix on the right will be $A^{-1}$.

**Example 1.10b** If possible find inverse of given matrices.

## 1.6 Matrix equations

A ***matrix equations*** is an equation in which a variable stands for a matrix.

**Remarks**

- $A \cdot X = B \quad \Leftrightarrow \quad X = A^{-1} \cdot B$
- $X \cdot A = B \quad \Leftrightarrow \quad X = B \cdot A^{-1}$
- $A \cdot X \cdot B = C \quad \Leftrightarrow \quad X = A^{-1} \cdot C \cdot B^{-1}$

**Example 1.11** Given $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 1 \\ 0 & -3 \end{bmatrix}$ determine matrices $X$ and $Y$ satisfying:

**1)** $A \cdot X = B,$ **b)** $Y \cdot A = B,$ **c)** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cdot X = I.$

## 1.7* Matrix algebra in macroeconomics

Macroeconomics is concerned with the working of the economy as a whole rather than with individual markets.

**Example 1.12.** A macroeconomic model of the economy.

**Symbols and assumptions:**

1. Households considered in aggregate earn their incomes (in a form of wages, salaries or profits) by producing output.
   $Q$ - value of output
   $Y$ – aggregate household income
2. Aggregate household income must equal the value of output: $Y = Q$.
3. Output must be bought by somebody, otherwise it would not be produced so value of output must equal demand (or aggregate expenditure): $Q = E$, where $E$ – demand; aggregate expenditure.
4. Aggregate expenditure consists of consumption expenditure and investments expenditure: $E = C + I$, where $C$– consumption expenditure, $I$– investment expenditure (assumed exogenous).
5. Planed household consumption $\tilde{C}$ is a (linear) function of household income: $\tilde{C} = aY + b$, where $a$ is called the marginal propensity to consume and $0 < a < 1, b > 0$. $a > 0$ means increase in planned consumption.
6. Planed or desired savings $\tilde{S} = Y - \tilde{C} = Y - (aY + b) = (1 - a)Y - b$ (savings function), where $1 - a$ is the marginal propensity to save.
7. Taxes $T$ of all income is at a rate $t$: $T = tY$, so disposable income is: $Y_d = Y - T$.
8. $\tilde{G}$ is (government) spending on goods and services.

**Identities of the model:**

$E = Y = C + I + \tilde{G}$ (equilibrium condition) **(1)**

$C = \tilde{C} = aY_d + b = a(Y - T) + b$ (consumption function, a behavioural relationship) **(2)**

$T = tY$ **(3)**

$$\begin{cases} (1) \\ (2) \\ (3) \end{cases} \Leftrightarrow \begin{cases} Y - C = I + \tilde{G} \\ -aY + C + aT = b \\ -tY + T = 0 \end{cases} \Leftrightarrow \begin{bmatrix} 1 & -1 & 0 \\ -a & 1 & a \\ -t & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} Y \\ C \\ T \end{bmatrix} = \begin{bmatrix} I + \tilde{G} \\ b \\ 0 \end{bmatrix}$$

The matrix form: $A \cdot x = B$, where:

$A = \begin{bmatrix} 1 & -1 & 0 \\ -a & 1 & a \\ -t & 0 & 1 \end{bmatrix}$ matrix of parameters (the marginal propensity to consume $a$ and tax rate $t$)

$x = \begin{bmatrix} Y \\ C \\ T \end{bmatrix}$ is a vector of unknows (income $Y$, consumption $C$, taxes $T$)

9. $B = \begin{bmatrix} I + \tilde{G} \\ b \\ 0 \end{bmatrix}$ is a vector of exogenous variables (investment $I$, spending on goods and services $\tilde{G}$)

$$A \cdot x = B \Leftrightarrow x = A^{-1} \cdot B \Leftrightarrow \begin{bmatrix} Y \\ C \\ T \end{bmatrix} = \frac{1}{1 - a(1 - t)} \begin{bmatrix} 1 & 1 & -a \\ a(1 - t) & 1 & -a \\ t & t & 1 - a \end{bmatrix} \cdot \begin{bmatrix} I + \tilde{G} \\ b \\ 0 \end{bmatrix}$$