Section 1 Calculus of one variable functions

1.1. Limit of a function

➢ Limit of a sequence

Let be given nonempty set $X \neq \emptyset$ and the set of natural numbers $\mathbb{N} = \{1, 2, 3, ...\}$.

Definition 1.1. An **infinite sequence** is a function whose domain is the set of all natural numbers:

$$a: \mathbb{N} \to X, \ \forall n \in \mathbb{N}: \ a: n \mapsto a(n) = a_n \in X.$$

Remarks

- the element a_n is called a **term of a sequence**; a_1 is the first term, a_2 the second term, and a_n the n-th term,
- if $X \subset \mathbb{R}$, then (a_n) is a **number sequence**.

Example 1.1.

$a_n = n$

n	1	2	3	4	5
a_n					

$b_n = \frac{1}{n}$

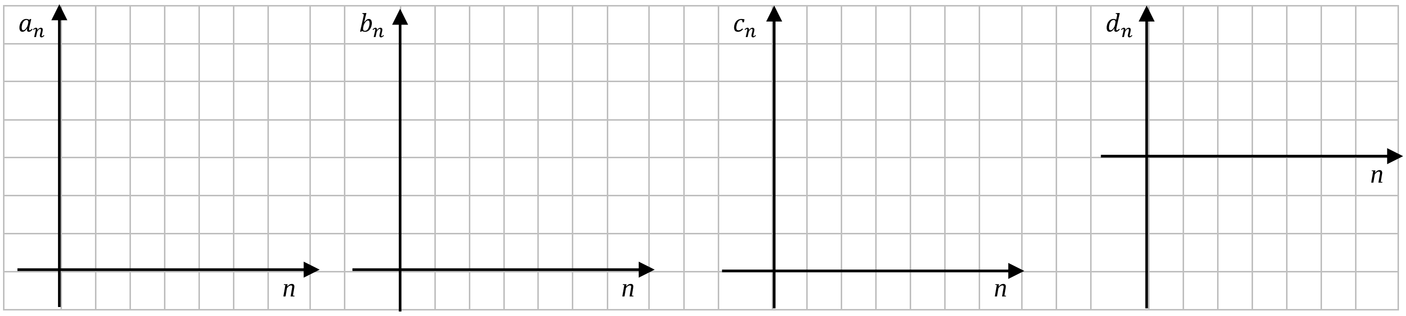
n	1	2	3	4	5
b_n					

$c_n = 2$

n	1	2	3	4	5
c_n					

$d_n = (-1)^n$

n	1	2	3	4	5
d_n					



$\lim_{n\to\infty} a_n = \cdots$ $\lim_{n\to\infty} b_n = \cdots$ $\lim_{n\to\infty} c_n = \cdots$ $\lim_{n\to\infty} d_n = \cdots$

$\lim_{x\to\infty} x = \cdots$ $\lim_{x\to\infty} \frac{1}{x} = \cdots$ $\lim_{x\to\infty} 2 = \cdots$

Remarks

- $\lim_{n\to\infty} a_n = L$ means that the number (or ∞) L is a limit of a sequence (a_n) as n tends to infinity.
- $\lim_{x\to\infty} f(x) = L$ means that L is a limit of a function f as x tends to infinity (∞).

Remark Not all sequences are quite as well behaved as convergent or divergent sequences. Some sequences do neither of those things. For instance, the sequence $a_n = (-1)^n$ does not converge or diverge. It just oscillates. An *oscillating* sequence may be bounded or unbounded. The sequence above is bounded because it oscillates between the bounds $[-1,1]$.

Elementary functions:

1. A *linear function*: $f(x) = ax + b,\ x \in \mathbb{R}$ (a *slope-intercept form*), where $a, b \in \mathbb{R}$.

2. A *quadratic function*: $f(x) = ax^2 + bx + c$, where $a \in \mathbb{R}\backslash\{0\},\ b, c \in \mathbb{R}$.

3. A *polynomial function* of degree n (n-th degree polynomial) in one variable x:

$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $n \in \mathbb{N}$, $a_n, a_{n-1}, \dots, a_1, a_0 \in \mathbb{R}$, $a_n \neq 0$, $x \in \mathbb{R}$.

4. A *rational function* (the quotient of two polynomials): $R(x) = \frac{P(x)}{S(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$

5. A *homographic function*: $f(x) = \frac{ax+b}{cx+d}$, where $ad - bc \neq 0$, $c \neq 0$, $x \in \mathbb{R}$.

6. **Trigonometric functions*:

- Sine function $f(x) = \sin x$.
- Cosine function: $f(x) = \cos x$.
- Tangent function: $f(x) = \tan x = \mathrm{tg} x$.
- Cotangent function: $f(x) = \cot x = \mathrm{ctg} x = \mathrm{ctn} x$.

7. An *exponential function* (with a base a): $f(x) = a^x,\ a \in (0,1) \cup (1, +\infty)$, $x \in \mathbb{R}$.

8. A *logarithmic function* (logarithm function) to the base a: $f(x) = \log_a x,\ x \in \mathbb{R}$,
where $a \in (0,1) \cup (1, +\infty),\ x \in (0, +\infty)$.

More information about elementary functions:

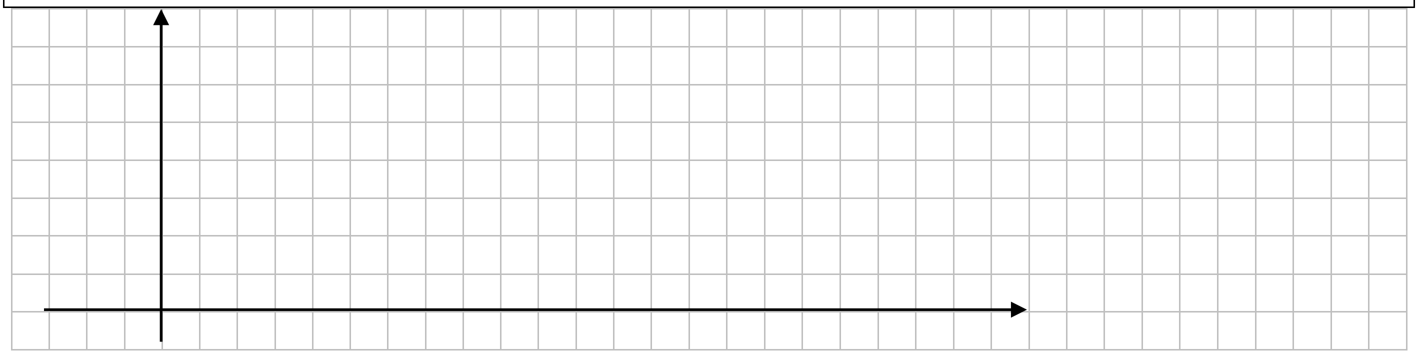
B. Ciałowicz - Workouts in Calculus and Linear Algebra with Applications to Economics

➢ Limit of function

Consider the function $f\colon D_f \to \mathbb{R}$, $D_f \subset \mathbb{R}$, $x_0 \in \mathbb{R}$.

Definition 1.2. (Heini's/sequential definition of limit of function)
The number L is a limit of a function f as x tends to (approaches) x_0 : $\lim\limits_{x \to x_0} f(x) = L$

$$\lim_{x \to x_0} f(x) = L \overset{H}{\Leftrightarrow} \forall (x_n)\colon \left[\left(x_n \neq x_0\ \wedge\ x_n \in D_f\ \wedge\ \lim_{n \to \infty} x_n = x_0\right) \Rightarrow \lim_{n \to \infty} f(x_n) = L\right].$$



Remarks:

- $\lim\limits_{x \to +\infty} f(x) = L$ means that the number L is a limit of a function f as x tends to infinity.
- $\lim\limits_{x \to x_0} f(x) = L$ means that the number L is a limit of a function f as x tends to (approaches) x_0.
- If $L \in \mathbb{R}$ (is finite), then the limit is called **proper.**
- If $L = +\infty$ or $L = -\infty$, then the limit is **improper.**
- Definition1.2. is valid for $L = \pm\infty$ or $x_0 = \pm\infty$.

Theorem 1.1. (limits of exemplary functions)

1. $\lim_{x\to\infty} a\cdot x = \begin{cases} -\infty, & a<0 \\ +\infty, & a>0 \end{cases}$

2. $\lim_{x\to\infty} x^a = +\infty, \ a>0$

3. $\lim_{x\to\infty} \frac{1}{x^a} = 0$ dla $a>0$

4. $\lim_{x\to\infty} a^x = \begin{cases} 0, & a\in(0,1) \\ 1, & a=1 \\ +\infty, & a>1 \end{cases}$

5. $\lim_{x\to-\infty} a^x = \begin{cases} 0, & a>1 \\ 1, & a=1 \\ +\infty, & a\in(0,1) \end{cases}$

6. $\lim_{x\to\infty} \sqrt[x]{a} = 1$ for $a>0$

7. $\lim_{x\to 0}\frac{\sin x}{x} = 1, \quad \lim_{x\to 0}\frac{\mathrm{tg}\, x}{x} = 1$

8. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e \quad (e = 2{,}718281828459\ldots)$

$e - Neper's\ \ number$

Theorem 1.2. (properties of limits)

I. If $\lim_{x\to x_0} f(x) = a \neq 0, \ \lim_{x\to x_0} g(x) = +\infty$, then:

a) $\lim_{x\to x_0}(f(x)\pm g(x)) = [a\pm(+\infty)] = \pm\infty$

b) $\lim_{x\to x_0}(f(x)\cdot g(x)) = [a\cdot(+\infty)] = \begin{cases} -\infty, & a<0 \\ +\infty, & a>0 \end{cases}$

c) $\lim_{x\to x_0}\frac{f(x)}{g(x)} = \left[\frac{a}{+\infty}\right] = 0$

d) $\lim_{x\to x_0}\frac{f(x)}{g(x)} = \left[\frac{+\infty}{a}\right] = \begin{cases} -\infty, & a<0 \\ +\infty, & a>0 \end{cases}$

II. If $\lim_{x\to x_0} f(x) = +\infty, \ \lim_{x\to x_0} g(x) = +\infty$, then:

a) $\lim_{x\to x_0}(f(x)+g(x)) = [(+\infty)+(+\infty)] = +\infty$

b) $\lim_{x\to x_0}(f(x)\cdot g(x)) = [(+\infty)\cdot(+\infty)] = (+\infty)$

III. If $\lim_{x\to x_0} f(x) = a \neq 0, \ \lim_{n\to\infty} g(x) = 0$, then:

a) $\lim_{x\to x_0}\frac{f(x)}{g(x)} = \left[\frac{a}{0}\right] = \pm\infty$

b) $\lim_{x\to x_0}\frac{g(x)}{f(x)} = \left[\frac{0}{a}\right] = 0$

Indeterminate forms: $\left[\frac{0}{0}\right], \left[\frac{\infty}{\infty}\right], [0\cdot(\infty)], [\infty-\infty], \ [1^\infty], [0^0], [(\infty)^0].$

Example 1.2. Calculate the limits:

1) $\lim_{x\to 2}(2x^3-4x^2+5x-7) =$

2) $\lim_{x\to 2}\frac{x^2-4}{x+1} =$

3) $\lim_{x\to 2}\frac{x^2-4}{x-2} =$

4) $\lim_{x\to\infty}(4x^3-6x^2+7x-104) =$

5) $\lim_{x\to-\infty}(4x^3-6x^2+7x-104) =$

6) $\lim_{x\to 2}\frac{log_2 x}{x+1}$

7) $\lim_{x\to-\infty}\frac{e^x}{3x+5}$

8) $\lim_{x\to\infty}\frac{x^2-4}{x+1} =$

9) $\lim_{x\to 1}\frac{3}{x-1} =$

Theorem 1.3. The limit $\lim_{x\to x_0} f(x)$ exists iff one-side limits $\lim_{x\to x_0^-} f(x)$ and $\lim_{x\to x_0^+} f(x)$ exist and be the same:

$$\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = \lim_{x\to x_0} f(x).$$

Remarks

- If $\lim_{x\to x_0^+} f(x) \neq \lim_{x\to x_0^-} f(x)$, then the limit $\lim_{x\to x_0} f(x)$ does not exist !
- One needs to calculate one-side limits, if:
  a) domain of a function is the interval of the form $(a, +\infty)$ or $(-\infty, b)$ or (a, b) (where $a, b \in \mathbb{R}$),
  b) formula of a function changes in the neighbourhood of the point,
  c) calculations lead to the symbol $\left[\frac{a}{0}\right]$ (a – nonzero constant).

1.2. Continuity

A function whose graph is unbroken is said to be **continuous**. A gap or break in the graph of a function is called a **discontinuity**.

Definition 1.3. A function $f: D \to \mathbb{R}$, $D \subset \mathbb{R}$ is **continuous at the point** x_0 if and only if:

1) $x_0 \in D$,
2) $\lim_{x\to x_0} f(x)$ exists ($\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = \lim_{x\to x_0} f(x)$),
3) $\lim_{x\to x_0} f(x) = f(x_0)$.

Example 1.3. Determine whether the function $f(x) = \begin{cases} 3x+2, \text{ dla } & x<0 \\ -e^x+3, \text{ dla } & 0 \le x \le 3 \\ \frac{x+1}{x-3}, \text{ dla } & x>3 \end{cases}$

is continuous at the points $x_0 = 0$ and $x_1 = 3$.

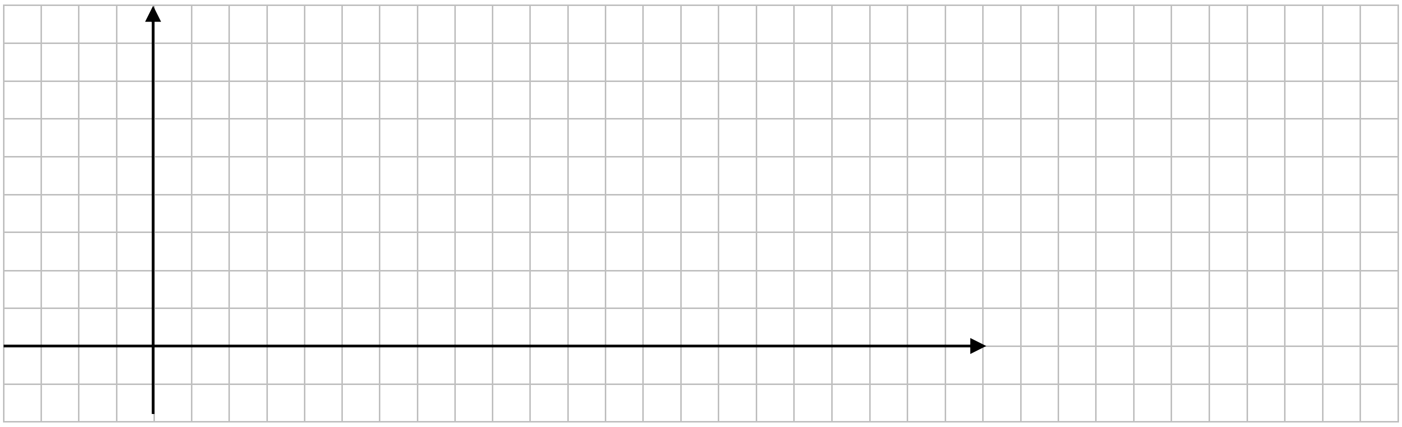
1.3. Derivative of function and its geometric interpretation

Let $f: D \to \mathbb{R}$, $D \subset \mathbb{R}$, $x_0, x \in D$.

$\Delta x = x - x_0$ denotes an increment (change) of the argument

$\Delta f = f(x) - f(x_0)$ denotes an increment (change) of the value

$\frac{\Delta f}{\Delta x} = \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ is called the **difference quotient**



Definition 1.4. A derivative of the function $\boldsymbol{f}$ at the point $\boldsymbol{x_0}$ is a limit of differential quotient $\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ as $\boldsymbol{\Delta x}$ approaches zero $\lim\limits_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$, if such a limit exists and is finite.

Notation: $f'(x_0) = \lim\limits_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$.

Remarks

- Other notations for the derivative of a function f at the point x_0:

$$f'(x_0) = Df(x_0) = \frac{df}{dx}(x_0) = \frac{dy}{dx}(x_0) = \dot{y}(x_0) \qquad \text{where} \quad y = f(x).$$

- Function which has a derivative at the point x_0 is called **differentiable at the point** x_0.
- Function f is differentiable iff the derivative $f'(x_0)$ exists at each point $x_0 \in D$.
- The function $f': D_{f'} \to \mathbb{R}$, such that $f': x \longmapsto f'(x)$ is called **a derivative of f**.
- If there exist the proper one-side limits of the differential quotient of function f, then we call them **one-side derivatives** and:

$$f'_+(x_0) = \lim_{\Delta x \to 0^+} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \quad \text{– the right-side derivative,}$$

$$f'_-(x_0) = \lim_{\Delta x \to 0^-} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \quad \text{– the left-side derivative.}$$

Example 1.4. Calculate the derivative of $f(x) = x^2$ at any point $x_0 \in \mathbb{R}$.

Theorem 1.4. Let $f: D \to \mathbb{R}$, $D \subset \mathbb{R}$ and $x_0 \in D$.
If the function f is differentiable at the point x_0, then it is continuous at this point.

Remark The inverse implication doesn't hold. The counterexample is an absolute value function: $f(x) = |x|$.

Theorem 1.5. (general rules/properties of derivative)
Let $f, g: D \to \mathbb{R}$ be differentiable functions. Then:

1) $[a \cdot f(x)]' = a \cdot f'(x), a \in \mathbb{R}$

2) $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$ (*the sum rule*)

3) $[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$ (*the product rule*)

4) $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$ $if \quad g(x) \neq 0$ (*the quotient rule*)

5) $(f \circ g)'(x_0) = \left[f\left(g(x_0)\right)\right]' = f'\left(g(x_0)\right) \cdot g'(x_0)$

a) $\left[ln\left(f(x)\right)\right]' = \dfrac{f'(x)}{f(x)}$ **b**) $\left[e^{f(x)}\right]' = e^{f(x)} f'(x)$ **c**) $\left[\sqrt{f(x)}\right]' = \dfrac{f'(x)}{2\sqrt{f(x)}}$

6) $\left[f(x)^{g(x)}\right]' = \left[e^{g(x)\ln f(x)}\right]' = f(x)^{g(x)} \cdot \left(g'(x) \cdot \ln f(x) + g(x) \cdot \frac{f'(x)}{f(x)}\right)$

Remark. A function $y = f(x)^{g(x)}$, where $f(x) > 0\ \forall x \in D_f$ is an **exponential-power function**.

Theorem 1.6. (special rules/derivatives of elementary functions)

1) $[c]' = 0$ for $c \in \mathbb{R}$

2) $[x^a]' = ax^{a-1}$, where $a \in \mathbb{R}$

3) $[a^x]' = a^x \ln a$, where $a > 0$

4) $[e^x]' = e^x$

5) $[\log_a x]' = \frac{1}{x \ln a}$, where $a > 0, a \neq 1$

6) $[\ln x]' = \frac{1}{x}$

7) $[\sin x]' = \cos x$

8) $[\cos x]' = -\sin x$

9) $[\tan x]' = \frac{1}{\cos^2 x}$

10) $[\cot x]' = -\frac{1}{\sin^2 x}$

Example 1.5. Find the domain and calculate derivative of the function:

1) $f(x) = 3x^2 + \frac{1}{x^2} - 5 + 4sinx - \log x$

2) $f(x) = e^x \cdot \sin x$

3) $f(x) = \frac{lnx}{5x^2 - 2x}$

4) $f(x) = e^{4x^3 - 5x}$

5) $f(x) = \ln(x^3 + 2x^2)$

6) $f(x) = \sqrt{4 - x^2}$

7) $h(x) = x^{sinx}$

➢ Geometric interpretation of a derivative

Secant is the line joining two given points $(x_0, f(x_0))$ and $(x_0 + \Delta x, f(x_0 + \Delta x))$ on the graph of a function. The slope of secant is given by the formula: $\frac{\Delta y}{\Delta x} = \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$.

Tangent line (simply: a **tangent**) to a given curve $f(x)$ at a point x_0 is a straight line which passes through the point $(x_0, f(x_0))$ and indicates the direction of the curve. A tangent line has a slope $f'(x_0)$.

Tangent line formula: $y = f'(x_0)(x - x_0) + f(x_0)$

Example 1.6. Find the equation of the line tangent to the graph of the function $f(x) = x^3$ when $x_0 = 1$ or $x_0 = 0$.

➢ Derivatives of higher orders

Definition 1.5. **A derivative of second order** (briefly: a second derivative) of a function f is a derivative of its first derivative: $f''(x) = [f'(x)]'$.
In general, **a derivative of order n** (briefly: an n-th derivative) of a function f is a derivative of $(n-1)$-th derivative of f: $f^{(n)}(x) = \left[f^{(n-1)}(x)\right]'$.

Notation

- second derivative: $f''(x) = D^2 f(x) = \frac{d^2 f}{dx^2} = \frac{d^2 y}{dx^2} = y'' = \ddot{y}$ where $y = f(x)$
- n-th derivative: $f^{(n)}(x) = D^n f(x) = \frac{d^n f}{dx^n}(x) = y^{(n)}$ where $y = f(x)$

Example 1.7. Find the 5th derivative of the function $f(x) = 4x^5 - 3x^4 + 7x^2 - 11$.

$f'(x) =$ $f''(x) =$ $f'''(x) =$ $f^{(4)}(x) =$ $f^{(5)}(x) =$

1.4. Differential of function and its applications

Definition 1.6. Let $f: D \to \mathbb{R}, D \subset \mathbb{R}, x_0 \in D$ and f be differentiable at x_0.
A differential of function f **at the point** x_0 is a linear function of the form

$$d_{x_0} f(h) = f'(x_0) \cdot h = f'(x_0) \cdot (x - x_0).$$

By definition of the derivative: $f'(x_0) \cdot \Delta x \approx f(x_0 + \Delta x) - f(x_0)$,
hence: $d_{x_0} f(\Delta x) \approx f(x_0 + \Delta x) - f(x_0)$
or equivalently ($\Delta x = x - x_0$):

$$f(x) = f(x_0 + \Delta x) \approx f(x_0) + d_{x_0} f(\Delta x) = f(x_0) + f'(x_0) \cdot \Delta x$$

In that easy way the formula for an **approximate value of function at a point** was derived.

Example 1.8. Using the differential find an approximate value of $\sqrt{4{,}02}$.

1.5. De L'Hospital's rule

Theorem 1.7. (l'Hospital's rule) Let $f, g: D \to \mathbb{R}$ ($D \subset \mathbb{R}$) be differentiable in some neighbourhood of x_0.
If $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ gives one of the indeterminate symbols $\left[\frac{0}{0}\right]$ or $\left[\frac{\infty}{\infty}\right]$ then $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$

Indeterminate symbols: $\left[\frac{0}{0}\right], \left[\frac{\infty}{\infty}\right], [0 \cdot (\infty)], [\infty - \infty], [1^{\infty}], [0^0], [(\infty)^0].$

Remarks

- The theorem above remains valid in cases of one-side limits: $x \to x_0^+$, $x \to x_0^-$ and limits in infinity: $x \to +\infty$, $x \to -\infty$.
- The theorem may be applied few times (if necessary). In problems involving repeated application of L'Hospital's rule, do not forget to verify that each new limit is of an indeterminate form before you apply the rule again.
- L'Hospital's rule involves differentiation of the numerator and the denominator separately.
- L'Hospital's rule applies only to quotients whose limits are indeterminate forms $\left[\frac{0}{0}\right]$ or $\left[\frac{\infty}{\infty}\right]$. Limits of the form $\frac{0}{\infty}$ or $\frac{\infty}{0}$, for example, are not indeterminate and L'Hospital's rule does not apply to such limits $\left(\frac{0}{\infty} = 0, \frac{\infty}{0} = \pm\infty\right)$.

Example 1.9. Find the limit:

1) $\lim_{x \to \infty} \frac{x^2 - 1}{2x^2 - 3x + 1} =$

2) $\lim_{x \to 0} \frac{e^x - 1}{x} =$

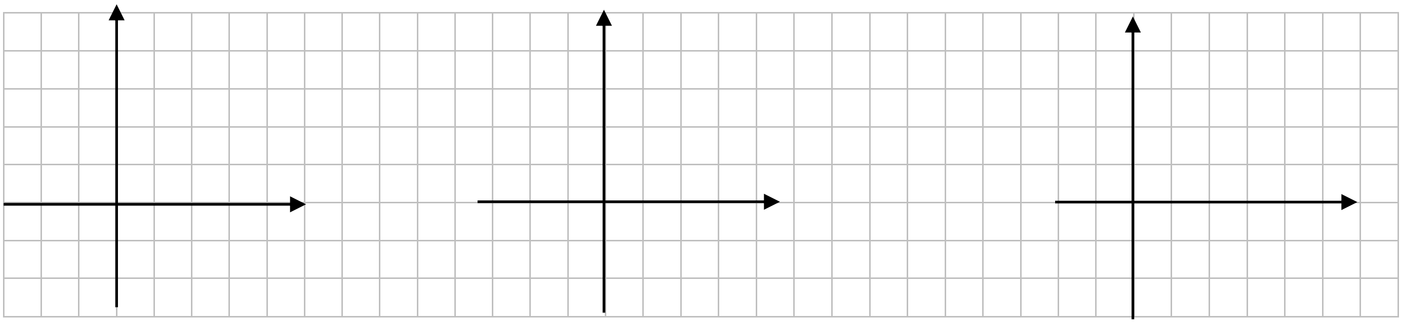
3) $\lim_{x \to \infty} \frac{e^x - 1}{x} =$

4) $\lim_{x \to 0} x^x =$

The symbol $[0 \cdot \infty]$: $\lim_{x \to x_0} f(x) \cdot g(x) = [0 \cdot \infty] = \lim_{x \to x_0} \frac{f(x)}{\frac{1}{g(x)}} = \left[\frac{0}{0}\right] = \lim_{x \to x_0} \frac{g(x)}{\frac{1}{f(x)}} = \left[\frac{\infty}{\infty}\right]$

The symbols $[0^0]$, $[\infty^0]$, $[1^{\infty}]$: $\lim_{x \to x_0} f(x)^{g(x)} = \lim_{x \to x_0} e^{g(x) \cdot \ln f(x)} = e^{\lim_{x \to x_0} g(x) \ln f(x)} = e^{[0 \cdot \infty]}$

1.6. Asymptotes of function



- **Vertical asymptotes**

Definition 1.7. Let $f\colon D_f \to \mathbb{R}$ and $x_0 \in \mathbb{R}$, $x_0 \notin D_f$.
A line $x = x_0$ is a **vertical asymptote** of the graph of function f iff at least one of the one-side limits of f at this point is improper:

a) if $\lim\limits_{x\to x_0^+} f(x) = \pm\infty$ then $x = x_0$ is **right-side vertical asymptote** of f at the point x_0

b) if $\lim\limits_{x\to x_0^-} f(x) = \pm\infty$ then $x = x_0$ is **left-side vertical asymptote** of f at the point x_0.

Example 1.10. Find vertical asymptotes of the graph of function:

1) $f(x) = \frac{x-2}{x^2-4}$

2) $f(x) = \frac{x}{lnx}$

- **Oblique asymptotes**

Definition 1.8. A straight line $y = ax + b$ is an **oblique (slant) asymptote** of the graph of f iff

$$\lim_{x\to+\infty}[f(x) - (ax + b)] = 0 \quad \text{or} \quad \lim_{x\to-\infty}[f(x) - (ax + b)] = 0.$$

a) If $\lim_{x\to+\infty}[f(x) - (ax + b)] = 0$ then $y = ax + b$ is a **right-side oblique asymptote** of f;

b) If $\lim_{x\to-\infty}[f(x) - (ax + b)] = 0$ then $y = ax + b$ is a **left-side oblique asymptote** of f;

c) If $a = 0$, then the asymptote has formula $y = b$ and is called a a **horizontal asymptote** (right-side, left-side or both-side, respectively).

Theorem 1.8.
A straight line $y = ax + b$ is an (**right-side/left-side**) **oblique asymptote** of the graph of f iff both the limits $a = \lim_{x\to\pm\infty}\frac{f(x)}{x}$ and $b = \lim_{x\to\pm\infty}(f(x) - ax)$ exist and are finite.

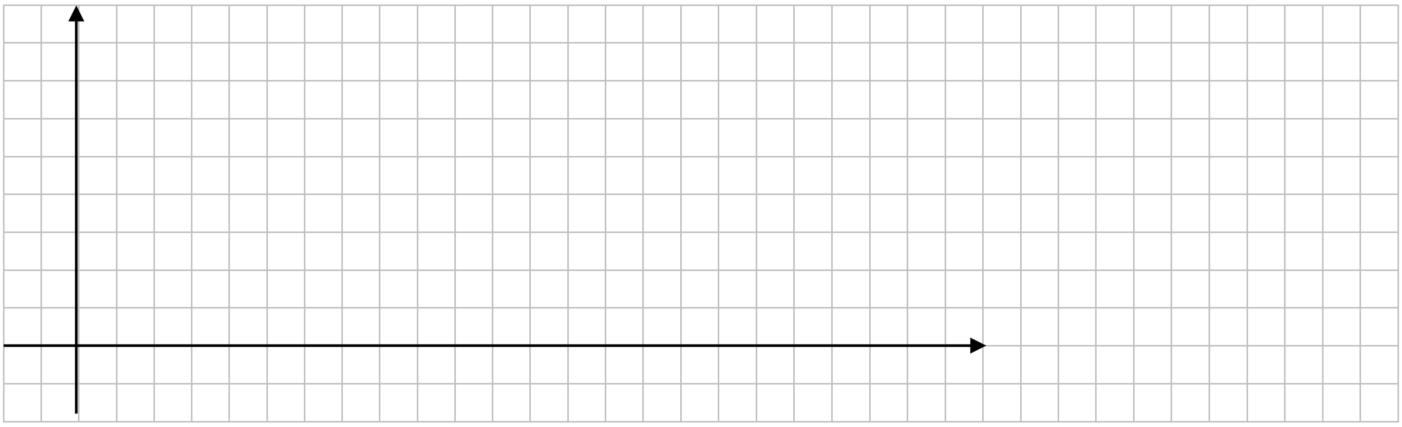
Example 1.11. Find oblique asymptotes of the graphs of functions:

1) $f(x) = \frac{3x^2+5x-2}{x-3}$

2) $f(x) = \frac{e^x}{2x-1}$

1.7. Increase and decrease (monotonicity). Relative (local) extrema.

Let $f: D_f \to \mathbb{R}$, $D_f \subset \mathbb{R}$, $[a,b] \subset D_f$.



Definition 1.9. A function f is called:

a) **increasing** ($f \nearrow$) on the interval (a,b) iff: $\forall x_1, x_2 \in (a,b)$ $[x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)]$

b) **decreasing** ($f \searrow$) on the interval (a,b) iff: $\forall x_1, x_2 \in (a,b)$ $[x_1 < x_2 \Longrightarrow f(x_1) > f(x_2)]$

c) **constant** on the interval (a,b) iff: $\forall x_1, x_2 \in (a,b)$ $[x_1 \neq x_2 \Longrightarrow f(x_1) = f(x_2)]$.

Theorem 1.9. (derivative test for intervals of increase and decrease)

Assume that the function f is continuous in $[a,b]$ and differentiable in (a,b). Then:

a) if $f'(x) > 0$ for $x \in (a,b)$, then f is increasing in (a,b),

b) if $f'(x) < 0$ for $x \in (a,b)$, then f is decreasing in (a,b),

c) if $f'(x) = 0$ for $x \in (a,b)$, then f is constant in (a,b).

Definition 1.10. Let $f: D \to \mathbb{R}$, $D \subset \mathbb{R}$, $x_0 \in D$. Function f has:

a) **a local maximum** at the point x_0 iff $\exists r > 0\ \forall x \in (x_0 - r, x_0 + r), x \neq x_0 :\ f(x) < f(x_0)$.

b) **a local minimum** at the point x_0 iff $\exists r > 0\ \forall x \in (x_0 - r, x_0 + r), x \neq x_0 :\ f(x) > f(x_0)$.

Remarks

- a local/relative maximum of a function is a peak, a point on the graph of the function that is higher than any neighboring point on the graph,
- a local/relative minimum of a function is the bottom of a valley, a point on the graph that is lower than any neighboring point,
- a local maximum need not be the highest point on the graph, it is maximal only relative to neighboring points; similarly, a local minimum need not to be the lowest point on the graph.

Theorem 1.10. (Fermat's/necessary condition for existence of extremum)

Let $f: D \to \mathbb{R}$, $D \subset \mathbb{R}$, $[a,b] \subset D$ $x_0 \in (a,b)$. Let f be diffirentiable in (a,b).

If the function f has an extremum at the point x_0, then $f'(x_0) = 0$.

Remarks

- Points in the domain of the function at which either $f'(x) = 0$ or the derivative is undefined are called *critical/stationary points*..
- The critical points are the possible relative maxima and minima of the function.

Theorem 1.11. (sufficient condition for the existence of extremum)
Assume that f is continuous at the point x_0 and differentiable in some its neighbourhood

1) If $f'(x) > 0$ for $x < x_0$ and $f'(x) < 0$ for $x > x_0$, then f has a **local maximum** at x_0.
2) If $f'(x) < 0$ for $x < x_0$ and $f'(x) > 0$ for $x > x_0$, then f assigns a **local minimum** at x_0.

Remark If under the assumption above the derivative does not change its sign, then the function doesn't have any extremum at this point.

How to use the first derivative to check monotonicity and find local extrema:

Step 1. Find the domain D_f of the given function $f(x)$.

Step 2. Compute the derivative $f'(x)$ and find its domain $D_{f'}$

Step 3. Find the values of x for which the derivative is zero or undefined.

Step 4. Determine where the function is increasing or decreasing by checking the sign of the derivative on the intervals whose endpoints are the values of x from Step 3.

Step 5. Summarize all observation.

Example 1.12. Find the intervals of increase and decrease and local extrema of the given function:

a) $f(x) = 2x - \ln x$

1. D_f

2. $f'(x) =$

3. $f'(x) = 0 \Leftrightarrow$

4. $f'(x) > 0 \Leftrightarrow$

$f'(x) < 0 \Leftrightarrow$

5.

$\boldsymbol{f'(x)}$				
$\boldsymbol{f(x)}$				

b) $f(x) = \frac{3e^x}{2-x}$

1. D_f

2. $f'(x) =$

3. $f'(x) = 0$

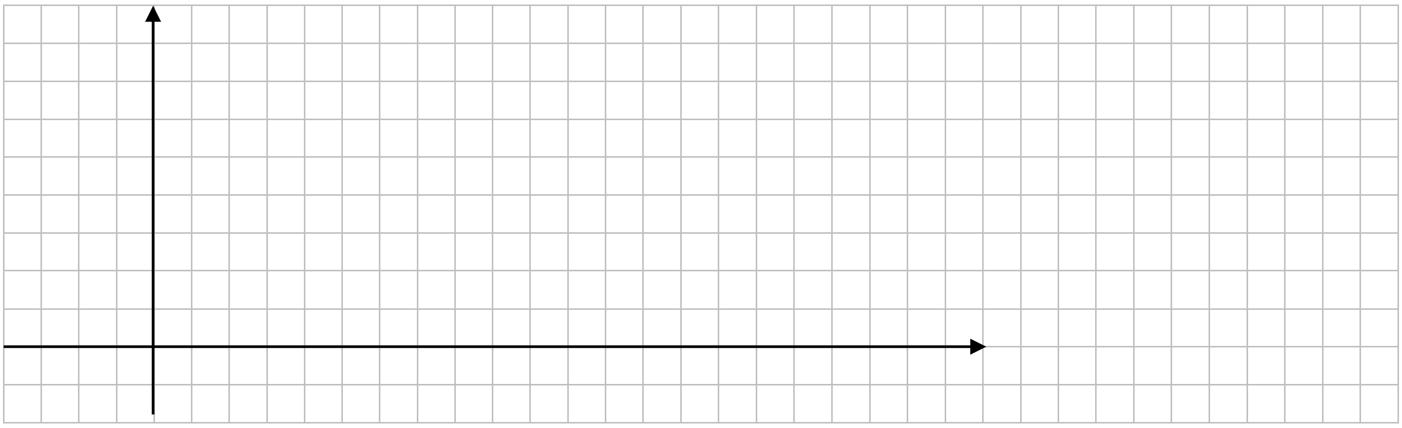
4. $f'(x) > 0 \Leftrightarrow$

$f'(x) < 0 \Leftrightarrow$

5.

$\boldsymbol{f'(x)}$						
$\boldsymbol{f(x)}$						

1.8. Concavity of function and inflection points



Remark Geometrically:

- if a function is **convex** in the interval, then at any point in it the tangent line lies under the graph of the function (touches it from below);
- if a function is **concave** in the interval, then at any point in it the tangent line lies above the graph of the function (touches it from above).

Definition 1.11. Let $\boldsymbol{f}: \boldsymbol{D} \to \mathbb{R}$, $\boldsymbol{D} \subset \mathbb{R}$ and $(\boldsymbol{a}, \boldsymbol{b}) \subset \boldsymbol{D}$.

1) The function f is **convex (convex downward, concave upward)** in the interval (a, b) iff a set $\{(x, y): x \in (a, b) \ \wedge \ y \geq f(x)\}$ is concave.

2) The function f is **concave (convex upward, concave downward)** in the interval (a, b) iff a set $\{(x, y): x \in (a, b) \ \wedge \ y \leq f(x)\}$ is concave.

3) Point $P = (x_0, f(x_0))$, where $x_0 \in D$ is called an **inflection point of the graph** of f iff f is convex (resp. concave) in some left-side neighbourhood of x_0 and is concave (resp. convex) in some right-side neighborhood of x_0.

Theorem 1.12. (second derivative test for concavity)
Let $f: D \to \mathbb{R}$, $D \subset \mathbb{R}$, $(a, b) \subset D$ and f is twice continuously differentiable in (a, b). Then:

a) if $\forall x \in (a, b): f''(x) > 0$ then f is convex in (a, b) $f \ \cup$

b) if $\forall x \in (a, b): f''(x) < 0$ then f is concave in (a, b) $f \ \cap$

Theorem 1.13. (necessary condition for existence of inflection point)
Let $f: D \to \mathbb{R}$, $D \subset \mathbb{R}$, $x_0 \in D$. Assume f is twice differentiable at x_0.
If $P(x_0, f(x_0))$ is an inflection point of the graph of f, then $f''(x_0) = 0$.

Remark. Inflection points can also occur at points in the domain of a function where the second derivative is undefined.

Theorem 1.14. (sufficient condition for existence of inflection point)
Let $f: D \to \mathbb{R}$, $D \subset \mathbb{R}$, $(a, b) \subset D$, $x_0 \in (a, b)$. Assume f is twice differentiable in the neighbourhood of x_0. If f'' is positive (resp. negative) in some left-side neighbourhood of x_0 and is negative (resp. positive) in some right-side neighbourhood of this point, then the point $(x_0, f(x_0))$ is an inflection point of the graph of f.

How to use the second derivative to check concavity and find inflection points:

Step 1. Find the domain D_f of the given function $f(x)$.

Step 2. Compute the first derivative $f'(x)$.

Step 3. Compute the second derivative $f''(x)$.

Step 4. Find the values of x for which the second derivative is zero (second-order critical points) or undefined.

Step 5. Determine where the function is concave or convex by checking the sign of the second derivative on the intervals whose endpoints are the values of x from Step 4.

Step 6. Summarize all observation.

Example 1.13. Test the following functions for the convexity. Find their inflection points.

a) $f(x) = 2x^2 + lnx$

1. $D_f =$

2. $f'(x) =$

3. $f''(x) =$

4. $f''(x) = 0 \Leftrightarrow$

5. $f''(x) < 0 \Leftrightarrow$

$f''(x) > 0 \Leftrightarrow$

6.

$f''(x)$							
$f(x)$							

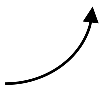
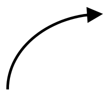
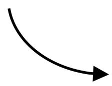
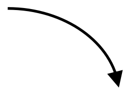
➢ Investigation of function

All the tools we've learnt up to now will allow us to investigate the properties of a function of one variable. Gathering them, it is possible to make a graph of such function. It will be our goal in this part.
The usual *scheme of investigation* of a function includes:

1. determining the domain;
2. calculating the characteristic points, i.e. zeros and points of graph intersection with coordinate system axes;
3. stating if the function is even/odd;
4. finding the asymptotes (if exist) – sometimes this step is called *investigation of function at the end-points of the domain*;
5. testing the function for monotonicity and extrema – for differentiable functions it is enough to use the first or the higher order derivative test, otherwise one should reason by definition, it is important to remember about domain of the derivative then;
6. testing the function for convexity and inflection points – as discussed, the second derivative test could be applied if the function is twice differentiable

It is convenient to summarize all the information in a table. Then making the graph of the function is immediate.

In the table the following notation will be used to agree the properties of monotonicity and convexity:

$f'(x)$	$+$	$+$	$-$	$-$
$f''(x)$	$+$	$-$	$+$	$-$
$f(x)$				

Example 1.14. Make a graph of: $f(x) = x \cdot e^{\frac{1}{x}}$

1.9. Economic applications

➢ Marginal analysis

Let $f: D \to \mathbb{R}$, $D \subset \mathbb{R}$, $x_0 \in D$ and f be differentiable at x_0.
Recalling that limit means approximate equality, by definition of the derivative one has:

$$f(x_0 + \Delta x) - f(x_0) \approx f'(x_0) \cdot \Delta x.$$

For $\Delta x = 1$:

$\boldsymbol{f(x_0 + 1) - f(x_0) \approx f'(x_0) = f_m(x)}$ - **marginality** of the function f at the point x_0

Interpretation: The derivative of function f at the point x_0 measures an approximate change of value of the function caused by one-unit increase in argument from the level x_0.

Example 1.15. Suppose the total cost of manufacturing x units of a certain commodity is $C(x)$, where: $C(x) = 3x^2 + 5x + 120, x \geq 0$.

a) Derive a formula for the marginal cost.
b) Find the marginal cost at $x = 10$ and interpret.
c) Find the level of manufacturing on which the average cost is equal to the marginal cost $\left(C_{av}(x) = \frac{C(x)}{x}\right)$.
d) Use the marginal cost to approximate the cost of producing the 51st unit.

➢ Elasticity of function

Percentage rate of change $= \frac{rate\ of\ change\ of\ quantity}{size\ of\ quantity}$

If $\boldsymbol{y = f(x)}$, then $\frac{f(x+\Delta x)-f(x)}{f(x)}$ **is called relative increment of the function f ,**

$\frac{\Delta x}{x}$ **is called relative increment of the argument x**

The percentage rate of change of y caused by one-percent increase in value of the argument x is given by the formula $\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{f(x_0)} : \frac{\Delta x}{x_0} = \frac{x_0}{f(x_0)} f'(x_0)$ and is called **elasticity of function** f **at the point** x_0:

$$E_x f(x_0) = \frac{x_0}{f(x_0)} \cdot f'(x_0)$$

Interpretation: This quantity describes a percentage change in value of the function f caused by one-percent increase in value of the argument x from the level x_0.

Levels of elasticity of demand

In general, the elasticity of demand $D(p)$ is negative, since demand decreases as price p increases.

- If $|E_p D(p)| > 1$, demand is said to be ***elastic*** with respect to price.
- If $|E_p D(p)| < 1$, demand is said to be ***inelastic*** with respect to price.
- If $|E_p D(p)| = 1$, demand is said to be ***unit elasticity*** with respect to price.

Example 1.16. Suppose the demand D and price p for a certain commodity are related by the equation: $D = 240 - 2p$ (for $0 \le p \le 120$).

a) Express the elasticity of demand as a function of p.
b) Calculate the elasticity of demand when the price is $p = 100$. Interpret the answer.
c) At what prices the elasticity of demand equal -1 ? What is the economic significant of this price?