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CFA Institute®
CFA Program

QUANTITATIVE METHODS

CFA® Program Curriculum
2025 • LEVEL II • VOLUME 1

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Quantitative Methods

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LEARNING MODULE

1

Basics of Multiple Regression and Underlying Assumptions

LEARNING OUTCOMES

Mastery	*The candidate should be able to:*
☐	describe the types of investment problems addressed by multiple linear regression and the regression process
☐	formulate a multiple linear regression model, describe the relation between the dependent variable and several independent variables, and interpret estimated regression coefficients
☐	explain the assumptions underlying a multiple linear regression model and interpret residual plots indicating potential violations of these assumptions

INTRODUCTION

Multiple linear regression uses two or more independent variables to describe the variation of the dependent variable rather than just one independent variable, as in simple linear regression. It allows the analyst to estimate using more complex models with multiple explanatory variables and, if used correctly, may lead to better predictions, better portfolio construction, or better understanding of the drivers of security returns. If used incorrectly, however, multiple linear regression may yield spurious relationships, lead to poor predictions, and offer a poor understanding of relationships.

The analyst must first specify the model and make several decisions in this process, answering the following, among other questions: What is the dependent variable of interest? What independent variables are important? What form should the model take? What is the goal of the model—prediction or understanding of the relationship?

The analyst specifies the dependent and independent variables and then employs software to estimate the model and produce related statistics. The good news is that the software, such as shown in Exhibit 1, does the estimation, and our primary tasks are to focus on specifying the model and interpreting the output from this software, which are the main subjects of this content.

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Exhibit 1: Examples of Regression Software

Software	Programs/Functions
Excel	Data Analysis > Regression
Python	scipy.stats.linregress
	statsmodels.lm
	sklearn.linear_model.LinearRegression
R	lm
SAS	PROC REG
	PROC GLM
STATA	regress

LEARNING MODULE OVERVIEW

- Multiple linear regression is used to model the linear relationship between one dependent variable and two or more independent variables.
- In practice, multiple regressions are used to explain relationships between financial variables, to test existing theories, or to make forecasts.
- The regression process covers several decisions the analyst must make, such as identifying the dependent and independent variables, selecting the appropriate regression model, testing if the assumptions behind linear regression are satisfied, examining goodness of fit, and making needed adjustments.
- A multiple regression model is represented by the following equation:

$$Y_i = b_0 + b_1X_{1i} + b_2X_{2i} + b_3X_{3i} + \ldots + b_kX_{ki} + \varepsilon_i,\ i = 1, 2, 3, \ldots, n,$$

  where Y is the dependent variable, Xs are the independent variables from 1 to k, and the model is estimated using n observations.
- Coefficient b_0 is the model's "intercept," representing the expected value of Y if all independent variables are zero.
- Parameters b_1 to b_k are the slope coefficients (or partial regression coefficients) for independent variables X_1 to X_k. Slope coefficient b_j describes the impact of independent variable X_j on Y, holding all the other independent variables constant.
- There are five main assumptions underlying multiple regression models that must be satisfied, including (1) linearity, (2) homoskedasticity, (3) independence of errors, (4) normality, and (5) independence of independent variables.
- Diagnostic plots can help detect whether these assumptions are satisfied. Scatterplots of dependent versus and independent variables are useful for detecting non-linear relationships, while residual plots are useful for detecting violations of homoskedasticity and independence of errors.

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USES OF MULTIPLE LINEAR REGRESSION

2

☐ describe the types of investment problems addressed by multiple linear regression and the regression process

There are many investment problems in which the analyst needs to consider the impact of multiple factors on the subject of research rather than a single factor. In the complex world of investments, it is intuitive that explaining or forecasting a financial variable by a single factor may be insufficient. The complexity of financial and economic relations calls for models with multiple explanatory variables, subject to fundamental justification and various statistical tests.

Examples of how multiple regression may be used include the following:

- A portfolio manager wants to understand how returns are influenced by a set of underlying factors; the size effect, the value effect, profitability, and investment aggressiveness. The goal is to estimate a Fama–French five-factor model that will provide an understanding of the factors that are important for driving a particular stock's excess returns.
- A financial adviser wants to identify whether certain variables, such as financial leverage, profitability, revenue growth, and changes in market share, can predict whether a company will face financial distress.
- An analyst wants to examine the effect of different dimensions of country risk, such as political stability, economic conditions, and environmental, social, and governance (ESG) considerations, on equity returns in that country.

Multiple regression can be used to identify relationships between variables, to test existing theories, or to forecast. We outline the general process of regression analysis in Exhibit 2. As you can see, there are many decisions that the analyst must make in this process.

For example, if the dependent variable is continuous, such as returns, the traditional regression model is typically the first step. If, however, the dependent variable is discrete—for example, an indicator variable such as whether a company is a takeover target or not a takeover target—then, as we shall see, the model may be estimated as a logistic regression.

In either case, the process of determining the best model follows a similar path. The model must first be specified, including independent variables that may be continuous, such as company financial features, or discrete (i.e., dummy variables), indicating membership in a class, such as an industry sector. Next, the regression model is estimated and analyzed to ensure it satisfies key underlying assumptions and meets the analyst's goodness-of-fit criteria. Once the model is tested and its out-of-sample performance is deemed acceptable, then it can be used for further identifying relationships between variables, for testing existing theories, or for forecasting.

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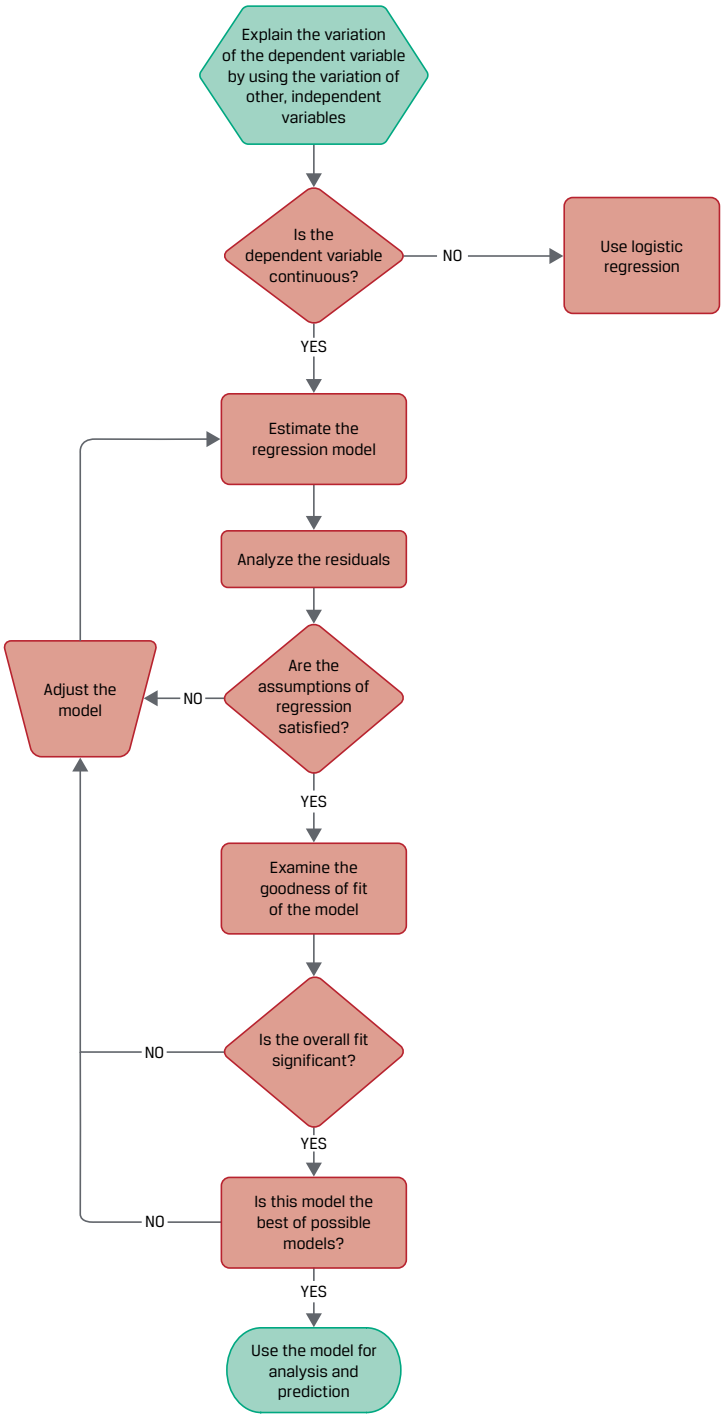
Exhibit 2: The Regression Process



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KNOWLEDGE CHECK

Assessment: Multiple Regression—Types of Investment Problems and Process

1. You are a junior analyst assisting in the development of various multiple regression models for your industry sector. Identify the action you should take to resolve each of the following issues:

Issue	Action
The dependent variable takes on a value of 1 if the company is a merger target and 0 otherwise.	
The analyst estimates a model with five independent variables, and none of these variables are significant explanatory variables.	
The residuals do not appear to be homoskedastic, thus violating a regression assumption.	
The regression assumptions are satisfied, the overall fit is significant, and the model is the best model of the possible models.	

Solution

Issue	Action
The dependent variable takes on a value of 1 if the company is a merger target and 0 otherwise.	Use logistic regression.
The analyst estimates a model with five independent variables, and none of these variables are significant explanatory variables.	Adjust the model and re-estimate.
The residuals do not appear to be homoskedastic, thus violating a regression assumption.	Adjust the model and re-estimate.
The regression assumptions are satisfied, the overall fit is significant, and the model is the best model of the possible models.	Use the model for analysis and prediction.

THE BASICS OF MULTIPLE REGRESSION

3

☐ formulate a multiple linear regression model, describe the relation between the dependent variable and several independent variables, and interpret estimated regression coefficients

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The goal of simple regression is to explain the variation of the dependent variable, *Y*, using the variation of an independent variable, *X*. The goal of multiple regression is the same, to explain the variation of the dependent variable, *Y*, but using the variations in a set of independent variables, $X_1, X_2, \ldots, X_k$. Recall the variation of *Y* is

$$\text{Variation of } Y = \sum_{i=1}^{n}(Y_i - \overline{Y})^2,$$

which we also refer to as the sum of squares total. The simple regression equation is

$$Y_i = b_0 + b_1X_i + \varepsilon_i,\ i=1, 2, 3, \ldots, n.$$

When we introduce additional independent variables to help explain the variation of the dependent variable, we have the multiple regression equation:

$$Y_i = b_0 + b_1X_{1i} + b_2X_{2i} + b_3X_{3i} + \ldots + b_kX_{ki} + \varepsilon_i,\ i = 1, 2, 3, \ldots, n. \quad (1)$$

In this equation, the terms involving the *k* independent variables are the deterministic part of the model, whereas the error term, ε_i, is the stochastic or random part of the model. The model is estimated over *n* observations, where *n* must be larger than *k*.

It is important to note that a slope coefficient in a multiple regression, known as a **partial regression coefficient** or a *partial slope coefficient*, must be interpreted with care. A partial regression coefficient, b_j, describes the impact of that independent variable on the dependent variable, holding all the other independent variables constant. For example, in the multiple regression equation,

$$Y_i = b_0 + b_1X_{1i} + b_2X_{2i} + b_3X_{3i} + \varepsilon_i,$$

the coefficient b_2 measures the change in *Y* for a one-unit change in X_2 assuming X_1 and X_3 are held constant. The estimated regression equation is

$$Y_i = \hat{b}_0 + \hat{b}_1X_{1i} + \hat{b}_2X_{2i} + \hat{b}_3X_{3i},$$

with ^ indicating estimated coefficients.

Consider an estimated regression equation in which the monthly excess returns of a bond index (RET) are regressed against the change in monthly government bond yields (BY) and the change in the investment-grade credit spreads (CS). The estimated regression, using 60 monthly observations, is

$$\text{RET} = 0.0023 - 5.0585\text{BY} - 2.1901\text{CS}.$$

We learn the following from this regression:

1. The bond index RET yields, on average, 0.0023% per month, or approximately 0.028% per year, if the changes in the government bond yields and investment-grade credit spreads are zero.
2. The change in the bond index return for a given one-unit change in the monthly government bond yield, BY, is –5.0585%, holding CS constant. This means that the bond index has an empirical duration of 5.0585.
3. If the investment-grade credit spreads, CS, increase by one unit, the bond index returns change by –2.1901%, holding BY constant.
4. For a month in which the change in the credit spreads is 0.001 and the change in the government bond yields is 0.005, the expected excess return on the bond index is

$$\text{RET} = 0.0023 - 5.0585(0.005) - 2.1901(0.001) = -0.0252, \text{ or } -2.52\%.$$

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KNOWLEDGE CHECK

An institutional salesperson has just read the research report in which you estimated a regression of monthly excess returns on a portfolio, RETRF, against the Fama–French three factors:

- MKTRF, the market excess return;
- SMB, the difference in returns between small- and large-capitalization stocks; and
- HML, the difference in returns between value and growth stocks.

All returns are stated in whole percentages (that is, 1 for 1%), and the estimated regression equation is

RETRF = 1.5324 + 0.5892MKTRF + −0.8719SMB + −0.0560HML.

Before this salesperson meets with her client firm, she asks you to do the following regarding your estimated regression model:

1. Interpret the intercept.

   Solution

   If the market excess return, SMB, and HML are each zero, then we expect a return on the portfolio of 1.5324%.

2. Interpret each slope coefficient.

   Solution

   Each slope coefficient is interpreted assuming the other variables are held constant.

   • For MKTRF, if the market return increases by 1%, we expect the portfolio's return to increase by 0.5892%.

   • For SMB, if the size effect returns increase by 1%, we expect the portfolio's return to decrease by 0.8719%.

   • For HML, if the value effect returns increase by 1%, we expect the portfolio's return to decrease by 0.056%.

3. Calculate the predicted value of the portfolio's return if

   MKTRF = 1, SMB = 4, and HML = –2.

   Solution

   Given the expected values of the independent variables, the expected return on the portfolio is

   $R = 1.534 + 0.5892(1) - 0.8719(4) - 0.0560(-2) = -1.2524.$

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4 ASSUMPTIONS UNDERLYING MULTIPLE LINEAR REGRESSION

☐ explain the assumptions underlying a multiple linear regression model and interpret residual plots indicating potential violations of these assumptions

Before we can conduct correct statistical inference on a multiple linear regression model estimated using ordinary least squares (OLS), we need to know whether the assumptions underlying that model are met. Suppose we have n observations on the dependent variable, Y, and the independent variables, $X_1, X_2, \ldots, X_k$, and we want to estimate the model

$$Y_i = b_0 + b_1X_{1i} + b_2X_{2i} + b_3X_{3i} + \ldots + b_kX_{ki} + \varepsilon_i,\ i = 1, 2, 3, \ldots, n.$$

In simple regression, we had four assumptions that needed to be satisfied so that we could make valid conclusions regarding the regression results. In multiple regression, we modify these slightly to reflect the additional independent variables:

1. Linearity: The relationship between the dependent variable and the independent variables is linear.
2. Homoskedasticity: The variance of the regression residuals is the same for all observations.
3. Independence of errors: The observations are independent of one another. This implies the regression residuals are uncorrelated across observations.
4. Normality: The regression residuals are normally distributed.
5. Independence of independent variables:

   5a. Independent variables are not random.

   5b. There is no exact linear relation between two or more of the independent variables or combinations of the independent variables.

The independence assumption is needed to enable the estimation of the coefficients. If there is an exact linear relationship between independent variables, the model cannot be estimated. In the more common case of approximate linear relationships, which may be indicated by significant pairwise correlations between the independent variables, the model can be estimated but its interpretation is problematic. In empirical work, the assumptions underlying multiple linear regression do not always hold. The statistical tools to detect violations and methods to mitigate their effects will be addressed later.

Regression software produces diagnostic plots, which are a useful tool for detecting potential violations of the assumptions underlying multiple linear regression. To illustrate the use of such plots, we first estimate a regression to analyze 10 years of monthly total excess returns of ABC stock using the Fama–French three-factor model. As noted previously, this model uses market excess return (MKTRF), size (SMB) and value (HML) as explanatory variables.

$$\text{ABC\_RETRF}_t = b_0 + b_1\text{MKTRF}_t + b_2\text{SMB}_t + b_3\text{HML}_t + \varepsilon_t$$

We start our analysis by generating a **scatterplot matrix** using software. This matrix is also referred to as a *pairs plot*.

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CODE: SCATTERPLOT MATRIX

Using Python

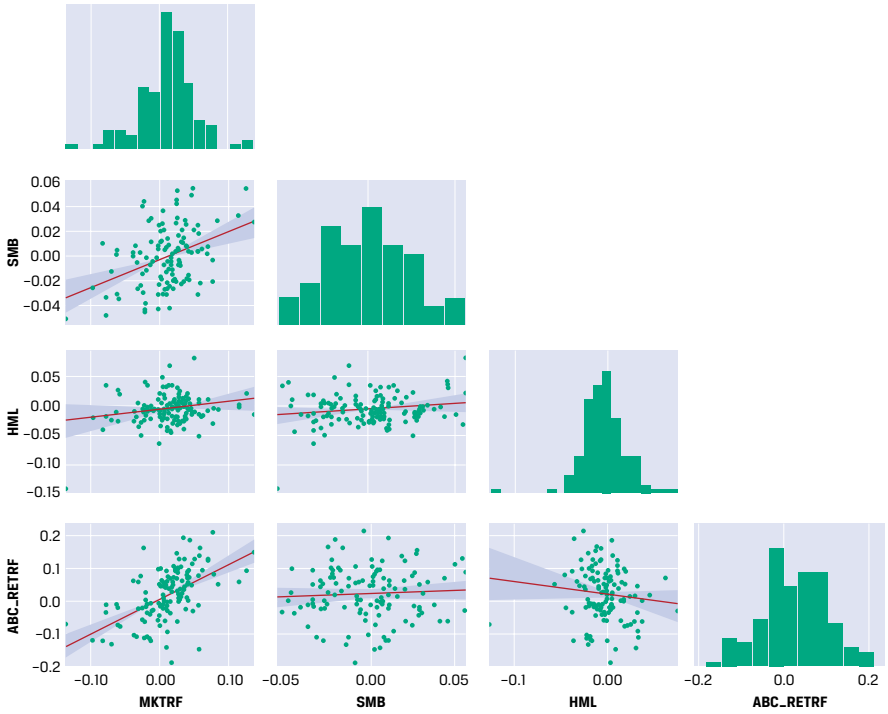
```
import pandas as pd
import matplotlib.pyplot as plt
import seaborn as sns
df = pd.read_csv("ABC_FF.csv",parse_dates=True,index_col=0)
sns.pairplot(df)
plt.show()
```

Using R

```
df <- read.csv("data.csv")
pairs(df[c("ABC_RETRF","MKTRF","SMB","HML")])
```

The pairwise scatterplots for all variables are shown in Exhibit 3. For example, the bottom row shows the relationships for the following three pairs: ABC_RETRF and MKTRF, ABC_RETRF and SMB, and ABC_RETRF and HML. The simple regression line and corresponding 95% confidence interval for the variables in each pair are also shown, along with the histogram of each variable along the diagonal.

Exhibit 3: Scatterplot Matrix of ABC Returns and Fama–French Factors



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You can see the following from the lower set of scatterplots between ABC_RET and the three independent variables:

- There is a positive relationship between ABC_RETF and the market factor, MKTRF.
- There seems to be no apparent relation between ABC_RETRF and the size factor, SMB. The reason is the scatterplot compares the two variables in isolation and does not show the "partial" correlation picked up by the regression, which explains why SMB is significant in the regression (see Exhibit 4) but not in the scatterplot.
- There is a negative relationship between ABC_RETF and the value factor, HML.

Looking at the scatterplots between the independent variables, SMB and HML have little or no correlation, as indicated by the relatively flat line for the SMB–HML pair. This is a desirable characteristic between explanatory variables.

An additional benefit of the scatterplot matrix is that all data points are displayed, so it can also be used to identify extreme values and outliers.

We now estimate the model of ABC's excess returns using software such as Microsoft Excel, Python, or R; results are shown in Exhibit 4. Focusing on the regression residuals, we look for clues to potential violations of the assumptions of multiple linear regression.

Exhibit 4: ABC Returns Explained Using Fama–French Three-Factor Model

Regression Statistics

Multiple *R*	0.6238
R-Squared	0.3891
Adjusted *R*-Squared	0.3733
Standard Error	0.0628
Observations	120

ANOVA

	Df	*SS*	*MS*	*F*	*Significance F*
Regression	3	0.2914	0.0971	24.6278	0.0000
Residual	116	0.4575	0.0039		
Total	119	0.7489			

	Coefficient	Standard error	*t*-Stat.	*P*-value	Lower 95%	Upper 95%
Intercept	0.0052	0.0061	0.8435	0.4007	–0.0070	0.0173
MKTRF	1.2889	0.1538	8.3791	0.0000	0.9842	1.5935
SMB	–0.5841	0.2664	–2.1922	0.0304	–1.1118	–0.0564
HML	–-0.6810	0.2231	–3.0523	0.0028	–1.1229	–0.2391

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CODE: REGRESSION

Using Python

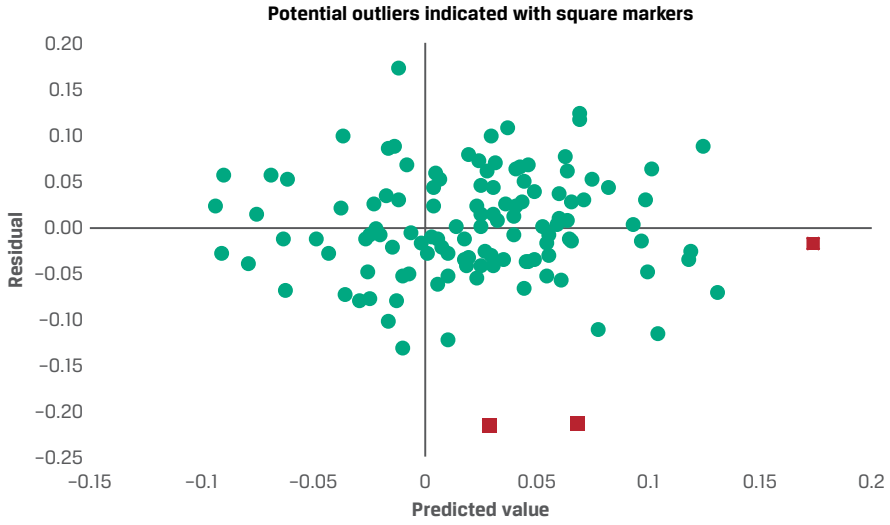
```
import pandas as pd
from statsmodels.formula.api import ols
df = pd.read_csv("data.csv")
model = ols('ABC_RETRF ~ MKTRF+SMB+HML',data=df).fit()
print(model.summary())
```

Using R

```
df <- read.csv("data.csv")
model <- lm('ABC_RETRF~ MKTRF+SMB+HML',data=df)
print(summary(model))
```

We start by looking at a scatterplot of residuals against the dependent variable, as shown in Exhibit 5. We can use this scatterplot to uncover potential assumption violations and to help identify outliers in our data.

Exhibit 5: Residuals vs. Predicted Value of Dependent Variable



Potential outliers indicated with square markers

As indicated by the line centered near residual value 0.00, a visual inspection of Exhibit 5 does not reveal any directional relationship between the residuals and the predicted values from the regression model. This outcome is good, because we want residuals to behave in an independent manner compared to what the model predicts, and suggests the regression's errors have a constant variance and are uncorrelated with each other, thereby satisfying several of the underlying assumptions of multiple linear regression.

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Notably, we detect three residuals (square markers) that may be outliers, Months 7, 25, and 95. This information can be used to check for shocks from factors not considered in the model that may have occurred at these points in time.

Exhibit 6 presents plots of the regression residuals versus each of the three factors in Panels A, B, and C. A visual inspection does not indicate any directional relationship between the residuals and the explanatory variables, suggesting there is no violation of a multiple linear regression assumption. Importantly, the three potential outliers detected in the residual versus predicted value plot are also apparent in Exhibit 6, as indicated by the square markers.

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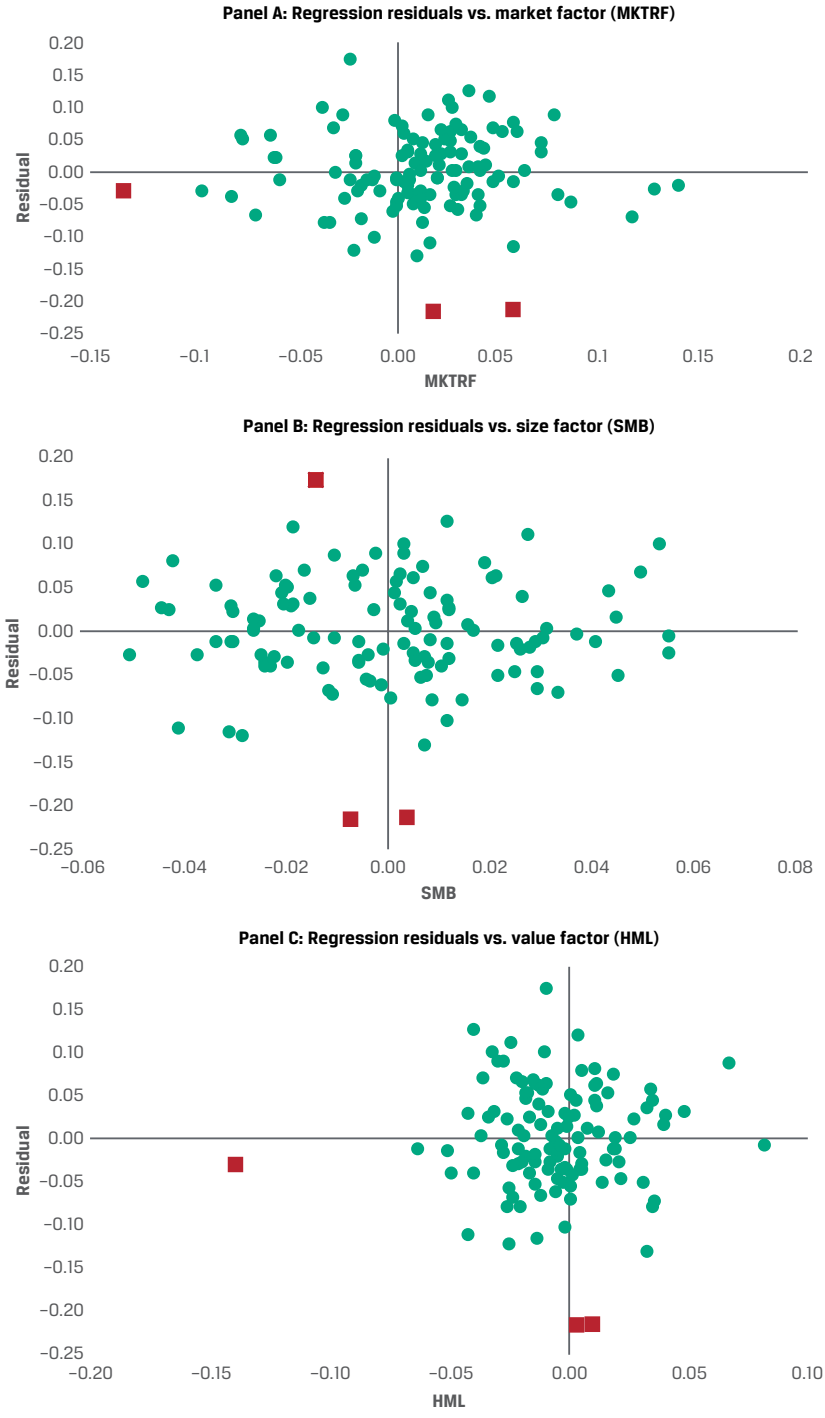
Exhibit 6: Regression Residuals vs. Factors (Independent Variables)



CODE: RESIDUAL ANALYSIS

Using Python

```
import pandas as pd
```

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```
import matplotlib.pyplot as plt

import statsmodels.api as sm

import numpy as np

df = pd.read_csv("data.csv",parse_dates=True,index_col=0)

model = ols('ABC_RETRF ~ MKTRF+SMB+HML',data=df).fit()

fig = sm.graphics.plot_partregress_grid(model)

fig.tight_layout(pad=1.0)

plt.show()

fig = sm.graphics.plot_ccpr_grid(model)

fig.tight_layout(pad=1.0)

plt.show()
```

Using R

```
library(ggplot2)

library(gridExtra)

df <- read.csv("data.csv")

model <- lm('ABC_RETRF~ MKTRF+SMB+HML',data=df)

df$res <- model$residuals

g1 <- ggplot(df,aes(y=res, x=MKTRF))+geom_point()+
xlab("MKTRF")+ylab("Residuals")

g2 <- ggplot(df,aes(y=res, x=SMB))+geom_point()+ xlab("SMB")+
ylab("Residuals")

g3 <- ggplot(df,aes(y=res, x=HML))+geom_point()+ xlab("HML")+
ylab("Residuals")

grid.arrange(g1,g2,g3,nrow=3)
```

Finally, in Exhibit 7 we present a **normal Q-Q plot**. A normal Q-Q plot, or simply a Q-Q plot, is used to visualize the distribution of a variable by comparing it to a normal distribution. In the case of regression, we can use a Q-Q plot to compare the model's standardized residuals to a theoretical standard normal distribution. If the residuals are normally distributed, they should align along the diagonal. Recall that 5% of observations that are normally distributed should fall below −1.65 standard deviations, so the 5th percentile residual observation should appear at −1.65 standard deviations.

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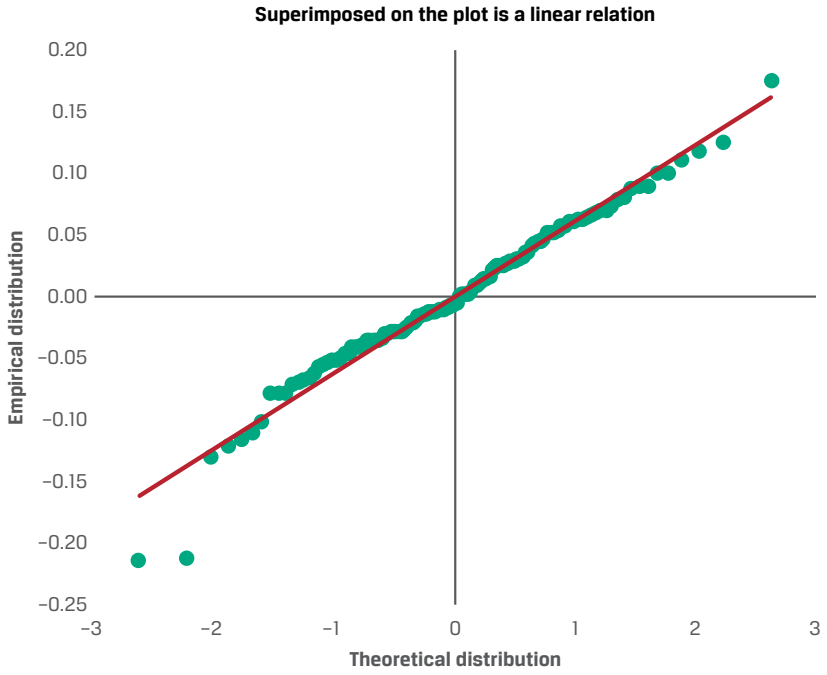
Exhibit 7: Normal Q-Q Plot of Regression Residuals



Superimposed on the plot is a linear relation

However, after –2 standard deviations, observations 25 and 95 fall well below the theoretical standard normal distribution range, while Observation 7, lying above the diagonal line around +2.5 standard deviations, is somewhat above the theoretical range. This evidence again suggests these three residual observations are potential outliers. However, setting them aside, the normal Q-Q plot does provide ample evidence that the regression residuals overall are distributed consistently with the normal distribution. Thus, we can conclude that the regression model error term is close to being normally distributed.

KNOWLEDGE CHECK

You are analyzing price changes of a cryptocurrency (CRYPTO) using the price changes for gold (GOLD) and a technology stock index (TECH), based on five years of monthly observations. You also run several diagnostic charts of your regression results. In a meeting with your research director, she asks you to do the following:

1. Identify any assumptions that may be violated if we examine the correlation between GOLD and TECH and find a significant pairwise correlation.

   Solution

   This result may indicate an approximate linear relation between GOLD and TECH, which would be a violation of the independence of independent variables, and should be explored further.

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2. Describe the purpose of a plot of the regression residuals versus the predicted value of CRYPTO.

Solution

This plot is useful for examining whether there is any clustering or pattern that may suggest the residuals are not homoskedastic and whether there are any potential outliers.

3. Describe the purpose of a plot of the regression residuals versus GOLD.

Solution

This plot is useful for examining whether there are any extreme values of the independent variables that may influence the estimated regression parameters and whether there is any relationship between the residuals and an independent variable, which suggests the model is misspecified.

4. Describe the purpose of a normal Q-Q plot of residuals.

Solution

The normal Q-Q plot is useful for exploring whether the residuals are normally distributed, a key assumption of linear regression.

5. A pairwise scatterplot is used to detect whether:

A. there is a linear relationship between the dependent and independent variables.

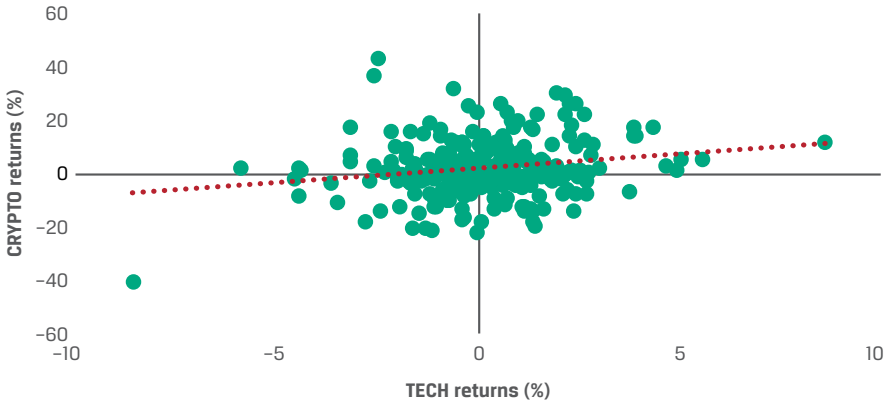
B. the residual terms exhibit heteroskedasticity.

C. the residual terms are normally distributed.

Solution

A is correct. The pairwise scatterplot is useful for visualizing the relationships between the dependent and explanatory variables.

6. Interpret this scatterplot showing price changes for the cryptocurrency (CRYPTO) and the tech index (TECH):



Solution

Based on the plot, there appears to be a positive relationship between CRYPTO and TECH, which may be significant. Several potential outliers are also apparent.

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7. A normal Q-Q plot is used to detect whether:

   A. there is a linear relationship between the dependent and independent variables.

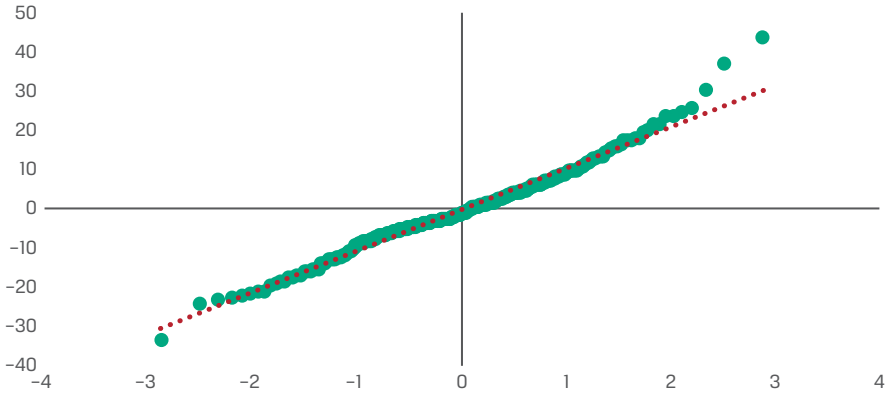
   B. the regression residual terms exhibit heteroskedasticity.

   C. the regression residual terms are normally distributed.

   Solution

   C is correct. The normal Q-Q plot is useful for exploring whether the residuals are normally distributed.

8. Interpret this normal Q-Q plot from our regression of CRYPTO price changes:



   Solution

   Based on the plot, the residuals are not normally distributed, as indicated by the deviation of residuals from the diagonal evident past ±2 standard deviations, and several potential outliers are also apparent. This normal Q-Q plot suggests the distribution of residuals is "fat-tailed." Note that fat-tailed distributions of residuals are a commonly observed feature of financial data time series.

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PRACTICE PROBLEMS

The following information relates to questions 1-5

You are a junior analyst at an asset management firm. Your supervisor asks you to analyze the return drivers for one of the firm's portfolios. She asks you to construct a regression model of the portfolio's monthly excess returns (RET) against three factors: the market excess return (MRKT), a value factor (HML), and the monthly percentage change in a volatility index (VIX).

You collect the data and run the regression, and the resulting model is

$Y_{RET} = -0.999 + 1.817X_{MRKT} + 0.489X_{HML} + 0.037X_{VIX}$.

You then create some diagnostic charts to help determine the model fit.

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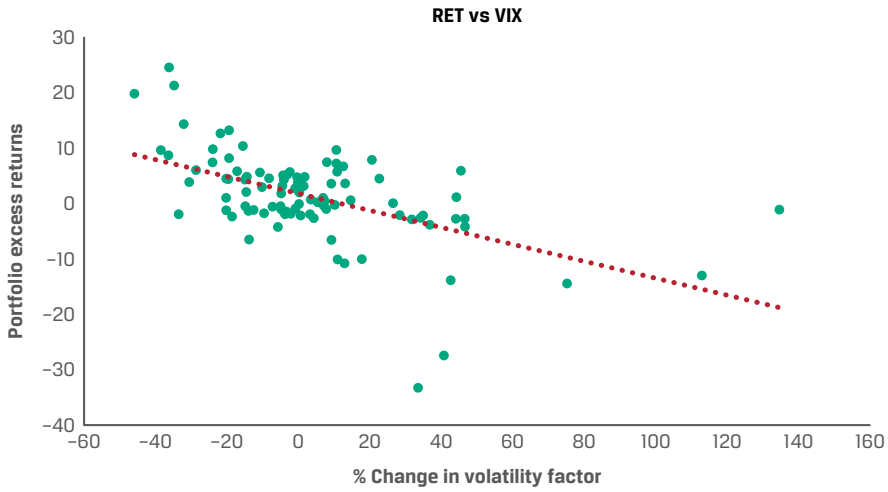

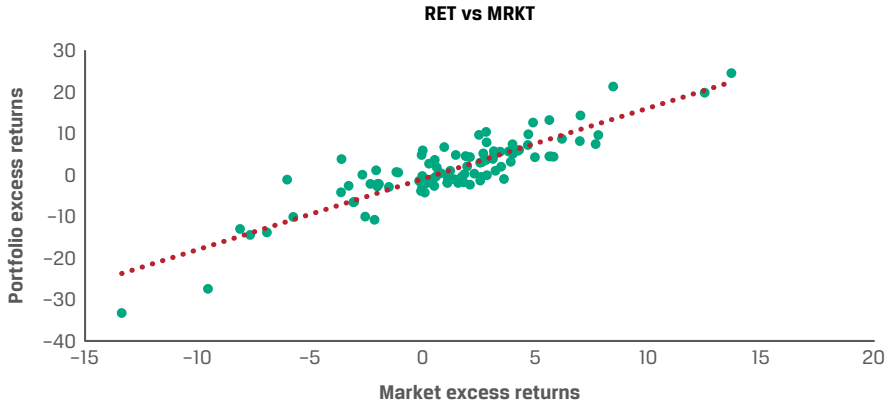

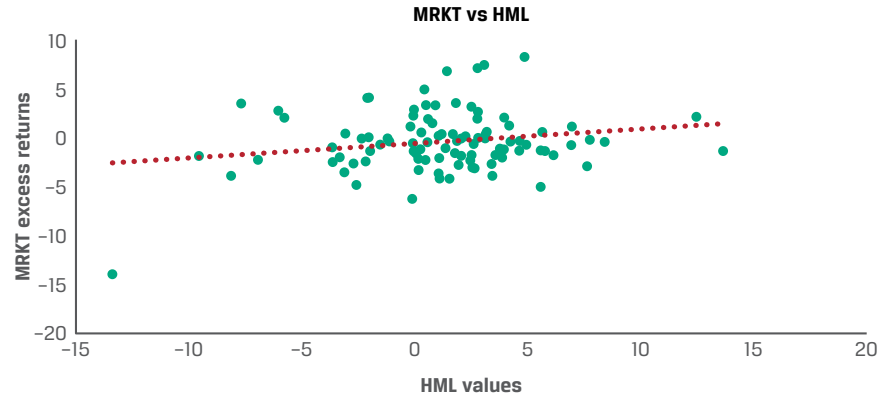

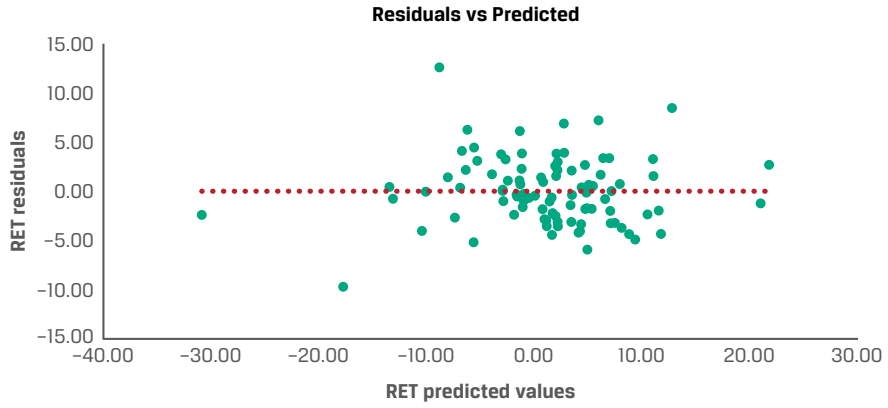

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1. Determine the type of regression model you should use.
   - **A.** Logistic regression
   - **B.** Simple linear regression
   - **C.** Multiple linear regression

2. Determine which one of the following statements about the coefficient of the volatility factor (VIX) is true.
   - **A.** A 1.0% increase in X_{VIX} would result in a –0.962% decrease in Y_{RET}.
   - **B.** A 0.037% increase in X_{VIX} would result in a 1.0% increase in Y_{RET}.
   - **C.** A 1.0% increase in X_{VIX}, holding all the other independent variables constant, would result in a 0.037% increase in Y_{RET}.

3. Identify the regression assumption that may be violated based on Chart 1, RET vs. VIX.
   - **A.** Independence of errors
   - **B.** Independence of independent variables
   - **C.** Linearity between dependent variable and explanatory variables

4. Identify which chart, among Charts 2, 3, and 4, is *most likely* to be used to assess homoskedasticity.
   - **A.** Chart 2
   - **B.** Chart 3
   - **C.** Chart 4

5. Identify which chart, among Charts 2, 3, and 4, is *most likely* to be used to assess independence of independent variables.
   - **A.** Chart 2
   - **B.** Chart 3
   - **C.** Chart 4

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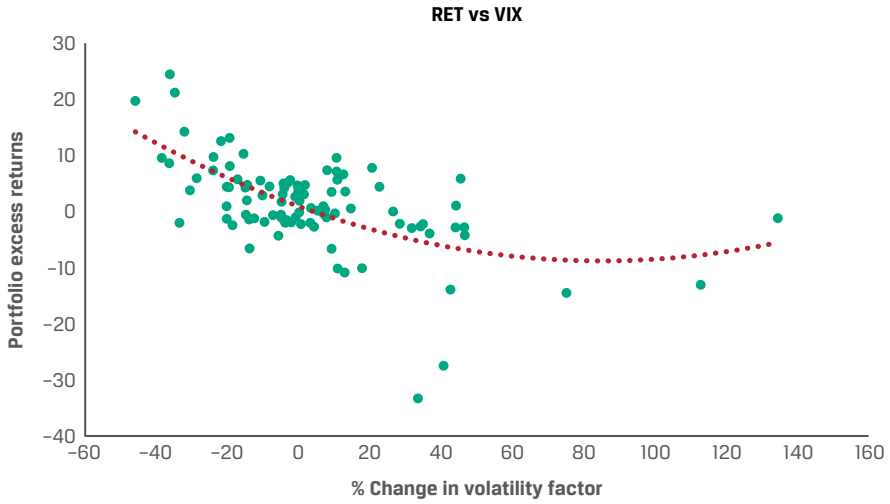
SOLUTIONS

1. C is correct. You should use a multiple linear regression model since the dependent variable is continuous (not discrete) and there is more than one explanatory variable. If the dependent variable were discrete, then the model should be estimated as a logistic regression.

2. C is correct. The coefficient of the volatility factor (X_{VIX}) is 0.037. It should be interpreted to mean that holding all the other independent variables constant, a 1% increase (decrease) would result in a 0.037% increase (decrease) in the monthly portfolio excess return (Y_{RET}).

3. C is correct. Chart 1 is a scatterplot of RET versus VIX. Linearity between the dependent variable and the independent variables is an assumption underlying multiple linear regression. As shown in the following Revised Chart 1, the relationship appears to be more curved (i.e., quadratic) than linear.



4. C is correct. To assess homoskedasticity, we must evaluate whether the variance of the regression residuals is constant for all observations. Chart 4 is a scatterplot of the regression residuals versus the predicted values, so it is very useful for visually assessing the consistency of the variance of the residuals across the observations. Any clusters of high and/or low values of the residuals may indicate a violation of the homoskedasticity assumption.

5. B is correct. Chart 3 is a scatterplot comparing the values of two of the independent variables, MRKT and HML. This chart would most likely be used to assess the independence of these explanatory variables.

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LEARNING MODULE

2

Evaluating Regression Model Fit and Interpreting Model Results

LEARNING OUTCOMES

Mastery	*The candidate should be able to:*
☐	evaluate how well a multiple regression model explains the dependent variable by analyzing ANOVA table results and measures of goodness of fit
☐	formulate hypotheses on the significance of two or more coefficients in a multiple regression model and interpret the results of the joint hypothesis tests
☐	calculate and interpret a predicted value for the dependent variable, given the estimated regression model and assumed values for the independent variable

INTRODUCTION 1

LEARNING MODULE OVERVIEW

- In multiple regression, adjusted R^2 is used as a measure of model goodness of fit since it does not automatically increase as independent variables are added to the model. Rather, it adjusts for the degrees of freedom by incorporating the number of independent variables.
- Adjusted R^2 will increase (decrease) if a variable is added to the model that has a coefficient with an absolute value of its t-statistic greater (less) than 1.0.
- Akaike's information criterion (AIC) and Schwarz's Bayesian information criteria (BIC) are also used to evaluate model fit and select the "best" model among a group with the same dependent variable. AIC is preferred if the purpose is prediction, BIC is preferred if goodness of fit is the goal, and lower values of both measures are better.
- Hypothesis tests of a single coefficient in a multiple regression, using t-tests, are identical to those in simple regression.

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- The joint *F*-test is used to jointly test a subset of variables in a multiple regression, where the "restricted" model is based on a narrower set of independent variables nested in the broader "unrestricted" model. The null hypothesis is that the slope coefficients of all independent variables outside the restricted model are zero.
- The general linear *F*-test is an extension of the joint *F*-test, where the null hypothesis is that the slope coefficients on all independent variables in the unrestricted model are equal to zero.
- Predicting the value of the dependent variable using an estimated multiple regression model is similar to that in simple regression. First, sum, for each independent variable, the estimated slope coefficient multiplied by the assumed value of that variable, and then add the estimated intercept coefficient.
- In multiple regression, the confidence interval around the forecasted value of the dependent variable reflects both model error and sampling error (from forecasting the independent variables); the larger the sampling error, the larger is the standard error of the forecast of *Y* and the wider is the confidence interval.

2 GOODNESS OF FIT

☐ evaluate how well a multiple regression model explains the dependent variable by analyzing ANOVA table results and measures of goodness of fit

In the simple regression model, the **coefficient of determination**, also known as *R*-squared or R^2, is a measure of the goodness of fit of an estimated regression to the data. R^2 can also be defined in multiple regression as the ratio of the variation of the dependent variable explained by the independent variables (sum of squares regression) to the total variation of the dependent variable (sum of squares total).

$$R^2 = \frac{\text{Sum of squares regression}}{\text{Sum of squares total}} = \frac{\sum_{i=1}^{n}\left(\hat{Y}_i - \bar{Y}\right)^2}{\sum_{i=1}^{n}\left(Y_i - \bar{Y}\right)^2},$$

where n is the number of observations in the regression, Y_i is an observation on the dependent variable, $\hat{Y}_i$ is the predicted value of the dependent variable based on the independent variables, and $\bar{Y}$ is the mean of the dependent variable.

In multiple linear regression, however, R^2 is less appropriate as a measure of a model's goodness of fit. This is because as independent variables are added to the model, R^2 will increase or will stay the same, but it will not decrease. Problems with using R^2 in multiple regression include the following:

- The R^2 cannot provide information on whether the coefficients are statistically significant.
- The R^2 cannot provide information on whether there are biases in the estimated coefficients and predictions.
- The R^2 cannot tell whether the model fit is good. A good model may have a low R^2, as in many asset-pricing models, and a bad model may have a high R^2 due to overfitting and biases in the model.

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Overfitting of a regression model is a situation in which the model is too complex, meaning there may be too many independent variables relative to the number of observations in the sample. A result of overfitting is that the coefficients on the independent variables may not represent true relationships with the dependent variable.

An alternative measure of goodness of fit is the **adjusted R^2** ($\overline{R}^2$), which is typically part of the multiple regression output produced by most statistical software packages. A benefit of using the adjusted R^2 is that it does not automatically increase when another independent variable is added to a regression. This is because it adjusts for the degrees of freedom as follows, where k is the number of independent variables:

$$\overline{R}^2 = 1 - \left(\frac{\text{Sum of squares error}/(n-k-1)}{\text{Sum of squares total}/(n-1)}\right). \quad (1)$$

Mathematically, the relation between R^2 and $\overline{R}^2$ is

$$\overline{R}^2 = 1 - \left[\left(\frac{n-1}{n-k-1}\right)(1-R^2)\right]. \quad (2)$$

Note that if $k \geq 1$, then R^2 is strictly greater than adjusted R^2. Further, the adjusted R^2 may be negative, whereas the R^2 has a minimum of zero.

The following are two key observations about $\overline{R}^2$ when adding a new variable to a regression:

- If the coefficient's t-statistic > |1.0|, then $\overline{R}^2$ increases.
- If the coefficient's t-statistic < |1.0|, then $\overline{R}^2$ decreases.

Note that a t-statistic with an absolute value of 1.0 does not indicate the independent variable is different from zero at typical levels of significance, 5% and 1%. So, adjusted R^2 does not set a very high bar for the statistic to increase.

Consider the regression output provided in Exhibit 1, which shows the results from the regression of portfolio returns on the returns for five hypothetical fundamental factors, which we shall call Factors 1 through 5. The goal of this regression is to identify the factors that best explain the returns on the portfolio.

Exhibit 1: Regression of Portfolio Excess Returns on Five Factors

Multiple R	0.7845
R-Squared	0.6155
Adjusted R-Squared	0.5718
Standard Error	0.0113
Log-Likelihood	-74.054
Observations	50

ANOVA Table

Source	Degrees of freedom	Sum of squares	Mean squares	*F*-statistic	Significance of *F*-statistic
Regression	5	90.6234	18.1247	14.0853	< 0.0000
Residual	44	56.6182	1.2868		
Total	49	147.2416			

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					95% confidence interval	
	Coefficient	**Standard error**	***t*-Statistic**	***P*-value**	**Lower bound**	**Upper bound**
Intercept	−2.1876	0.1767	−12.3787	0.0000	−2.5437	−1.8314
Factor 1	1.5992	0.2168	7.3756	0.0000	1.1622	2.0361
Factor 2	0.1923	0.7406	0.2596	0.7964	−1.3002	1.6847
Factor 3	−0.7126	0.5854	−1.2172	0.2300	−1.8925	0.4673
Factor 4	3.3376	1.3493	2.4736	0.0173	0.6182	6.0570
Factor 5	−2.6832	8.3919	−0.3197	0.7507	−19.5959	14.2295

CODE: REGRESSION STATISTICS

Using Microsoft Excel

Let depvar be the range of cells for the dependent variable, and let indvar be the range of cells for the independent variables.

```
=LINEST(depvar,indvar,TRUE,TRUE) or Data Analysis > Regression
```

Using Python

Let df be the data frame containing the data.

```
import statsmodels.api as sm

from statsmodels.stats.anova import anova_lm

from statsmodels.formula.api import ols

formula='Portfolio ~ Factor1+Factor2+Factor3+Factor4+Factor5'

results=ols(formula,df).fit()

print(results.summary())
```

Using R

Let df be the data frame containing the data.

```
model11 <- lm(df$Portfolio ~

df$Factor1+df$Factor2+df$Factor3+df$Factor4+df$Factor5)

anova(model11)

summary(model11)
```

We see in Exhibit 1 that R^2 is 0.6155, or 61.55%, and can we visualize this relationship using the graph in Exhibit 2.

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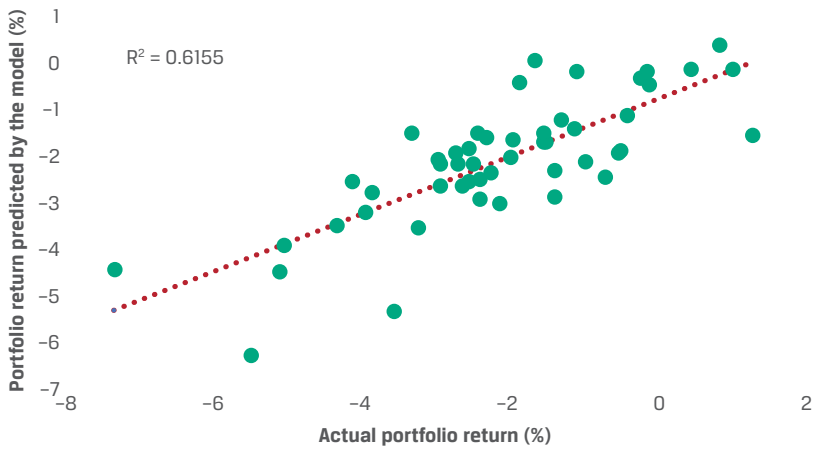
Exhibit 2: Predicted vs. Actual Portfolio Excess Returns Based on Regression of Returns on a Model with Five Factors



We can use the **analysis of variance (ANOVA)** table in Exhibit 1 to describe the model fit. We know from simple regression that the R^2 is the ratio of the sum of squares regression to the sum of squares total. We confirm this as

$$R^2 = \frac{\text{SSR}}{\text{SST}} = \frac{90.6234}{147.2416} = 0.6155$$

and the adjusted R^2 (using Equation 3) as

$$\overline{R}^2 = 1 - \left[\left(\frac{50-1}{50-5-1}\right)\left(1 - \frac{90.6234}{147.2416}\right)\right] = 0.5718.$$

The effect of successively adding each factor to the model is shown in Exhibit 3. The regression of the portfolio returns starts with the returns of Factor 1, then in the next model adds Factor 2, and so on, until all five are included in the full model. Note that with each added variable, the R^2 either stays the same or increases. However, while the adjusted R^2 increases when Factors 3 and 4 are added, it declines when Factors 2 and 5 are added to those models, respectively. This illustrates the relationship between the $|t$-statistic$|$ of the added variable and adjusted R^2.

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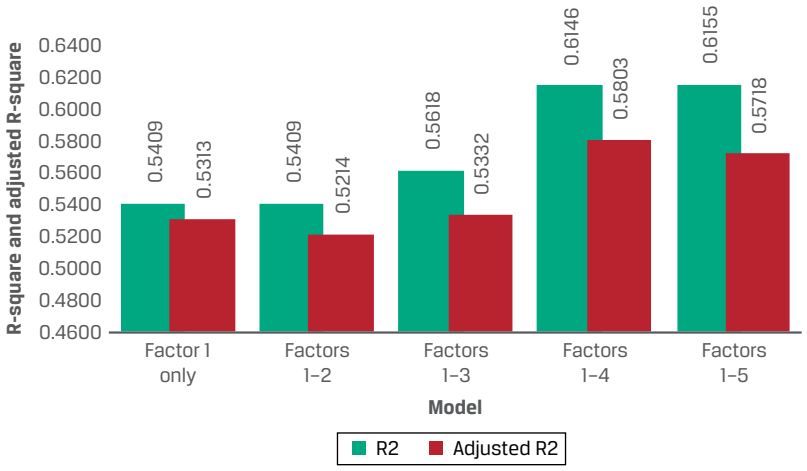
Exhibit 3: R^2 and Adjusted R^2 for Models Adding Factors to Explain Excess Returns



Importantly, the following should be noted:

- Unlike in simple regression, there is no neat interpretation of the adjusted R^2 in a multiple regression setting in terms of percentage of the dependent variable's variation explained.
- The adjusted R^2 does not address whether the regression coefficients are significant or the predictions are biased; this requires examining residual plots and other statistics.
- R^2 and adjusted R^2 are not generally suitable for testing the significance of the model's fit; for this, we explore the ANOVA further, calculating the F-statistic and other goodness-of-fit metrics.

KNOWLEDGE CHECK

You are a junior portfolio manager (PM) reviewing your firm's research on diversified manufacturers. You are considering Model 1, a cross-sectional regression of return on assets (ROA) for a sample of 26 diversified manufacturing companies on capital expenditures scaled by beginning year PPE (CAPEX):

Model 1: $\text{ROA}_i = b_0 + b_{CAPEX}\text{CAPEX}_i + \varepsilon_i$.

Multiple R	0.9380
R-Squared	0.8799
Adjusted R-Squared	0.8749
Standard Error	1.5274
Log-Likelihood	–46.842
Observations	26

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Partial ANOVA Results

					95% confidence interval	
	Coefficient	Standard error	*t*-Statistic	*P*-value	Lower bound	Upper bound
Intercept	4.7856	0.6468	7.3988	0.0000	3.4507	6.1206
CAPEX	1.2495	0.0942	13.2623	0.0000	1.0551	1.4440

Adding a second feature, the prior year's ratio of advertising expenditures to revenues (ADV), results in Model 2:

Model 2: $\text{ROA}_i = b_0 + b_{CAPEX}\text{CAPEX}_i + b_{ADV}\text{ADV}_i + \varepsilon_i$.

Multiple *R*	0.9384
R-Squared	0.8805
Adjusted *R*-Squared	0.8701
Standard Error	1.55666
Log-Likelihood	–46.795
Observations	26

Partial ANOVA Results

					95% confidence interval	
	Coefficient	Standard Error	*t*-Statistic	*P*-value	Lower 95%	Upper 95%
Intercept	4.9961	0.9144	5.4638	0.0000	3.1045	6.8876
CAPEX	1.2415	0.0990	12.5410	0.0000	1.0367	1.4463
ADV	–0.0345	0.1039	–0.3320	0.7429	–0.2495	0.1805

1. Interpret and contrast R^2 and adjusted R^2 for Models 1 and 2 using the regression output provided.

   Solution

   The R^2 for Model 1 (CAPEX only) indicates that 87.99% of the variation of ROA is explained by CAPEX. For Model 2 (CAPEX and ADV), the R^2 increases to 88.05%. However, the adjusted R^2 for Model 2 declines to 0.8701 (87.01%) from 0.8749 for Model 1. The lower adjusted R^2 is consistent with the |*t*-statistic| for ADV's coefficient < 1.0 (i.e., –0.3320) and the coefficient not being different from zero at typical significance levels (*P*-value = 0.7429). To conclude, adding the ADV variable does not improve the overall statistical performance and explanatory power of the model.

As both the R^2 and adjusted R^2 may increase when we add independent variables, we risk model overfitting. Fortunately, there are several statistics to help compare model quality and identify the most parsimonious model, including two statistics more commonly known by their acronyms, AIC and BIC.

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We can use **Akaike's information criterion (AIC)** to evaluate a collection of models that explain the same dependent variable. It is often provided in the output for regression software, but AIC can be calculated using information in the regression output:

$$\text{AIC} = n \ln\left(\frac{\text{Sum of squares error}}{n}\right) + 2(k+1). \quad (3)$$

As the formula indicates, the AIC statistic depends on the sample size (n), the number of independent variables in the model (k), and the sum of squares error (SSE) of the model. One goal in multiple regression is to derive the best fitting model without adding extraneous independent variables. AIC is a measure of model parsimony, so a lower AIC indicates a better-fitting model. The term $2(k + 1)$ is the penalty assessed for adding independent variables to the model.

In a similar manner, **Schwarz's Bayesian information criterion (BIC or SBC)** allows comparison of models with the same dependent variable, as follows:

$$\text{BIC} = n \ln\left(\frac{\text{Sum of squares error}}{n}\right) + \ln(n)(k+1). \quad (4)$$

Compared to AIC, BIC assesses a greater penalty for having more parameters in a model, so it will tend to prefer small, more parsimonious models. This is because $\ln(n)$ is greater than 2, even for very small sample sizes. Because we also use BIC to choose the best model among a set of models (i.e., the one with the lowest BIC), when do we prefer one measure over the other?

Practically speaking, AIC is preferred if the model is used for prediction purposes, but BIC is preferred when the best goodness of fit is desired. Importantly, the value of these measures considered alone is meaningless; the relative values of AIC or BIC among a set of models is what really matters.

For the regression of portfolio returns on five factors, we present several goodness-of-fit measures in Exhibit 4 generated for five models out of the 31 possible combinations ($_5C_r$, r = 1–5) of models using these five variables.

Exhibit 4: Goodness-of-Fit Measures for Portfolio Excess Returns Regressed on Different Sets of Factors

	R^2	Adjusted R^2	AIC	BIC
Factor 1 only	0.541	0.531	19.079	*22.903*
Factors 1 and 2	0.541	0.521	21.078	26.814
Factors 1, 2, and 3	0.562	0.533	20.743	28.393
Factors 1, 2, 3, and 4	0.615	0.580	*16.331*	25.891
Factors 1, 2, 3, 4, and 5	0.615	0.572	18.215	29.687

The following are important observations that can be made from Exhibit 4:

- The R^2 increases or remains the same as we add variables to the model.
- The adjusted R^2 increases with the addition of some variables (Factors 3 and 4) but decreases with the addition of other variables (Factors 2 and 5).
- The AIC is minimized with the model using Factors 1, 2, 3, and 4.
- The BIC is minimized with the model using Factor 1 only.

If we are developing a model for prediction purposes, then we would likely select the four-factor model that AIC indicates, whereas if we are seeking the most parsimonious, best-fitting model, we would choose the one-factor model. We now have a framework for selecting the best model from a given set of models.

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KNOWLEDGE CHECK

1. The research report you are reviewing presents goodness-of-fit statistics for the two models explaining the variation in return on assets (ROA) for the sample of diversified manufacturers, as follows:

	Model 1: CAPEX Only	*Model 2: CAPEX and ADV*
R^2	0.880	0.881
Adjusted R^2	0.875	0.870
AIC	23.899	25.804
BIC	26.523	28.792

The senior PM as asks you to interpret the given goodness-of-fit statistics and justify which is the better model.

Solution

The goodness-of-fit results for this sample regression indicate the following:

- 88% of the variation in ROA is explained by CAPEX, and 88.1% of this variation is explained by CAPEX and ADV together.
- Adjusted R^2 declines when ADV is added, indicating its coefficient is not significant and has a $|t\text{-statistic}| < 1.0$ (note that for ADV, the P-value = 0.7429 and the t-statistic is −0.3320).
- AIC and BIC both indicate that Model 1 is a better model as these metrics are lower for Model 1 than for Model 2.

To conclude, adding ADV does not improve the explanatory or predicting power of the original model using just CAPEX.

TESTING JOINT HYPOTHESES FOR COEFFICIENTS

3

☐ formulate hypotheses on the significance of two or more coefficients in a multiple regression model and interpret the results of the joint hypothesis tests

In multiple regression, the interpretation of the intercept and slope coefficients is similar to that in simple regression, but with a subtle difference. In simple regression, the intercept is the expected value of the dependent variable if the independent variable is zero. In multiple regression, the intercept is the expected value of the dependent variable if *all* independent variables are zero.

Regarding interpretation of slope coefficients, as noted earlier, in multiple regression, the slope coefficient for a given independent variable is the expected change in the dependent variable for a one-unit change in that independent variable *with all other independent variables remaining constant.*

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Tests of a single coefficient in a multiple regression are identical to those in a simple regression. The hypothesis structure is the same, and the *t*-test is the same. For a two-sided alternative hypothesis that the true coefficient, b_j, is equal to a hypothesized value, B_j, the null and alternative hypotheses are

H_0: $b_j = B_j$ and H_a: $b_j \neq B_j$,

where b_j is the coefficient on the *j*th independent variable and B_j is the hypothesized slope (0, 1, or something else). A one-sided test for a single coefficient is also the same in multiple regression as in simple regression:

One-sided coefficient test, left side	*One-sided coefficient test, right side*
H_0: $b_j \geq B_j$, H_a: $b_j < B_j$	H_0: $b_j \leq B_j$, H_a: $b_j > B_j$

If we are testing simply whether a variable is significant in explaining the dependent variable's variation, the hypotheses are H_0: $b_j = 0$ and H_a: $b_j \neq 0$.

By default, statistical software produces the *t*-statistics and the *P*-values for a test of the slope coefficient against zero for each independent variable in the model. If we want to test the slope against a hypothesized value other than zero, we need to

- perform the test by adjusting the hypothesized parameter value, B_j, in the test statistic or
- compare the hypothesized parameter value, B_j, with the confidence interval bounds for the coefficient generated in the regression output.

There are times when we want to test a subset of variables in a multiple regression jointly. Just to motivate the preliminary discussion and frame the problem, suppose we want to compare regression results for a portfolio's excess returns using Fama and French's three-factor model (MKTRF, SMB, HML) with those using their five-factor model (MKTRF, SMB, HML, RMW, CMA). Because the two models share three factors (MKTRF, SMB, HML), a comparison involves examining whether the two other variables—the return difference between the most profitable and the least profitable firms (RMW) and the return difference between firms that invest most conservatively and those that invest most aggressively (CMA)—are needed. A key objective in determining the better model is parsimony, achieved by identifying groups of independent variables that are most useful in explaining variation in the dependent variable.

Now consider a more general model:

$$Y_i = b_0 + b_1X_{1i} + b_2X_{2i} + b_3X_{3i} + b_4X_{4i} + b_5X_{5i} + \varepsilon_i.$$

We refer to this full model, with all five independent variables, as the **unrestricted model**. Suppose we want to test whether X_4 and X_5 together do not provide a significant contribution to explaining the dependent variable—that is, to test whether $b_4 = b_5 = 0$. We compare the full model with five independent variables to

$$Y_i = b_0 + b_1X_{1i} + b_2X_{2i} + b_3X_{3i} + \varepsilon_i.$$

This model is referred to as the **restricted model** because by excluding them from the model, we have restricted the slope coefficients on X_4 and X_5 to be equal to zero. These models are also described as **nested models**, because the restricted model is "nested" within the unrestricted model. This comparison of models implies a null hypothesis that involves a joint restriction on two coefficients—that is, H_0: $b_4 = b_5 = 0$ against H_A: b_4 and/or $b_5 \neq 0$.

We can use a statistic to compare nested models, where the unrestricted model is compared to a restricted model in which one or more of the slope coefficients is set equal to zero. This statistic focuses on the impact of the joint restriction on the ability of the restricted model to explain the dependent variable relative to the unrestricted model. We test the role of the jointly omitted variables using the following *F*-distributed test statistic:

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$$F = \frac{(\text{Sum of squares error restricted model} - \text{Sum of squares error unrestricted})/q}{\text{Sum of squares error unrestricted model}/(n-k-1)}, \quad (5)$$

where q is the number of restrictions, meaning the number of variables omitted in the restricted model compared to the unrestricted model.

Suppose we want to compare a model with five independent variables to a restricted model having only three of these variables (X_1, X_2, and X_3).

Unrestricted model: $Y_i = b_0 + b_1X_{1i} + b_2X_{2i} + b_3X_{3i} + b_4X_{4i} + b_5X_{5i} + \varepsilon_i$.

Restricted model: $Y_i = b_0 + b_1X_{1i} + b_2X_{2i} + b_3X_{3i} + \varepsilon_i$.

Here, $q = 2$ since we are testing the null hypothesis of $b_4 = b_5 = 0$. Also, note the F-statistic for this test has q and $n - k - 1$ degrees of freedom.

To summarize, the unrestricted model has the larger set of explanatory variables, while the restricted model has q fewer independent variables because the slope coefficients on the excluded variables are constrained to equal zero.

The general form for hypotheses for testing a nested (restricted) model is

H_0: $b_m = b_{m+1} = \ldots = b_{m+q-1} = 0$; H_a: At least one slope of the q slopes $\neq 0$,

where m is the first restricted slope, $m + 1$ is the second restricted slope, and so on, up to the qth restricted slope.

Why not simply perform hypothesis tests on the individual variables and then draw conclusions about the set from that information? Often in multiple regression involving financial variables, there is some degree of correlation between the variables, so there may be some sharing of explanatory power that is not considered with the tests of individual slopes.

We now apply this test to the model of portfolio returns with the five hypothetical factors (Factors 1–5), introduced in Exhibit 1, as independent variables. Partial ANOVA results for the restricted model with just Factors 1, 2, and 3 and for the unrestricted model with all five factors are shown in Exhibit 5, Panel A. The **joint test of hypotheses** for the slopes of Factors 4 and 5 using the F-test are in Panel B, using a 1% significance level. As demonstrated, we fail to reject the null hypothesis that the slopes of Factors 4 and 5 are both zero.

Exhibit 5: Comparison of Regression Models of Portfolio Excess Returns Using Three Factors and Five Factors (from Exhibit 1)

Panel A Partial ANOVA Results for Models Using Three Factors and Five Factors

Source	Factors	Degrees of freedom	Residual sum of squares	Mean square error
Restricted model	1, 2, 3	46	64.5176	1.4026
Unrestricted model	1, 2, 3, 4, 5	44	56.6182	1.2868

Panel B Test of Hypotheses for Factors 4 and 5 at 1% Level of Significance

Step 1	State the hypotheses.	H_0: $b_{Factor4} = b_{Factor5} = 0$ vs. H_a: At least one $b_j \neq 0$
Step 2	Identify the appropriate test statistic.	$F = \frac{(\text{Sum of squares error restricted model} - \text{Sum of squares error unrestricted})/q}{\text{Sum of squares error unrestricted model}/(n-k-1)}$, with $q = 2$ and $n - k - 1 = 44$ degrees of freedom.

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Step 3	Specify the level of significance.	α = 1% (one-tail, right side).
Step 4	State the decision rule.	Critical *F*-value = 5.120. Reject the null if the calculated *F*-statistic exceeds 5.120.
Step 5	Calculate the test statistic.	$F = \frac{(64.5176 - 56.6182)/2}{56.6182/44} = \frac{3.9497}{1.2868} = 3.0694$
Step 6	Make a decision.	Fail to reject the null hypothesis because the calculated *F*-statistic does not exceed the critical *F*-value. There is not sufficient evidence to indicate that at least one slope coefficient among b_4 and b_5 is different from zero.

This joint hypothesis test indicates Factors 4 and 5 do not provide sufficient explanatory power (i.e., SSE declines by just 7.8994 = 64.5176 – 56.6182) to compensate for the loss of two degrees of freedom by their inclusion in the unrestricted model. Thus, we conclude the restricted, more parsimonious model fits the data better than the unrestricted model.

CODE: COMPARING NESTED REGRESSION MODELS USING THE FIVE FACTORS (FROM EXHIBIT 1)

Python

Let df be the data frame containing the data.

```
from statsmodels.stats.anova import anova_lm

from statsmodels.formula.api import ols

formula='Portfolio ~ Factor1+Factor2+Factor3+Factor4+Factor5'

results=ols(formula,df).fit()

hypotheses='(Factor4=Factor5=0)'

f_test = results.f_test(hypotheses)

print(f_test)
```

R

Let df be the data frame containing the data.

```
model5 <- lm(df$Portfolio ~
df$Factor1+df$Factor2+df$Factor3+df$Factor4+df$Factor5)

model3 <- lm(df$Portfolio ~ df$Factor1+df$Factor2+df$Factor3)

anova(model5,model3)
```

We can extend the *F*-distributed joint test of hypotheses for coefficients to test the significance of the whole regression equation, which is often referred to as a goodness-of-fit test. For the multiple linear regression,

$$Y_i = b_0 + b_1X_{1i} + b_2X_{2i} + b_3X_{3i} + \ldots + b_kX_{ki} + \varepsilon_i,$$

where k is the number of independent variables, we can use the **general linear *F*-test** to test the null hypothesis that slope coefficients on all variables are equal to zero:

$$H_0: b_1 = b_2 = b_3 = \ldots = b_k = 0$$

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against the alternative that at least one slope coefficient is different from zero:

H_a: At least one $b_j \neq 0$.

This *F*-statistic is calculated in the same way as in simple regression—the ratio of the mean square regression (MSR) to the mean square error (MSE)—but the degrees of freedom are now k in the numerator and $n - k - 1$ in the denominator:

$$F = \frac{\text{MSR}}{\text{MSE}}.$$

Using the ANOVA table in Exhibit 1 for the five-factor model of portfolio returns,

$$F = \frac{18.1247}{1.2868} = 14.0853.$$

We present partial ANOVA results in Exhibit 6, Panel A, and show in Panel B the steps for the hypothesis test for model goodness of fit using the *F*-statistic at the 5% significance level.

Exhibit 6: Test of Hypothesis for Goodness of Fit Using *F*-Statistic for Model of Portfolio Returns Regressed against Five Factors

Panel A Partial ANOVA Results (repeated from Exhibit 1)

Analysis of Variance

Source	Degrees of freedom	Sum of squares	Mean squares
Regression	5	90.6234	18.1247
Residual	44	56.6182	1.2868
Total	49	147.2416	

Panel B Hypothesis Testing

Step 1	State the hypotheses.	H_0: $b_1 = b_2 = b_3 = b_4 = b_5 = 0$. H_a: At least one $b_j \neq 0$.
Step 2	Identify the appropriate test statistic.	$F = \frac{\text{MSR}}{\text{MSE}}$, with 5 and 44 degrees of freedom.
Step 3	Specify the level of significance.	α = 5% (one-tail, right side).
Step 4	State the decision rule.	Critical *F*-value = 2.427. Reject the null if the calculated *F*-statistic is greater than 2.427.
Step 5	Calculate the test statistic.	$F = \frac{18.1247}{1.2868} = 14.0853$.
Step 6	Make a decision.	Reject the null hypothesis because the calculated *F*-statistic exceeds the critical *F*-value. There is sufficient evidence to indicate that at least one slope coefficient is different from zero.

Exhibit 7 summarizes the statistics we have introduced for judging the goodness of fit of multiple regression models. Importantly, finding the "best" model is not a straight-line process but is, rather, iterative, because it depends on reviewing the regression results and adjusting the model accordingly.

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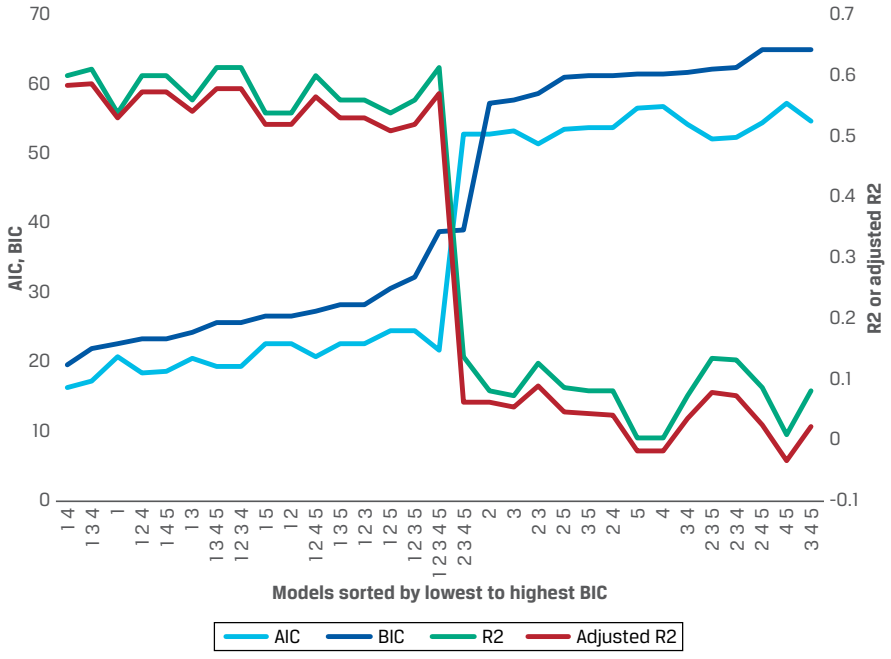
Exhibit 7: Assessing Model Fit Using Multiple Regression Statistics

Statistic	Criterion to use in assessment
Adjusted R^2	The higher the better
Akaike's information criterion (AIC)	The lower the better
Schwarz's Bayesian information criterion (BIC)	The lower the better
t-Statistic on a slope coefficient	Outside bounds of critical *t*-value(s) for the selected significance level
F-test for joint test of slope coefficients	Exceeds the critical *F*-value for the selected significance level

Exhibit 8 shows model fit statistics visually for all 31 possible models from our example of regressing portfolio returns on the five factors. Here the models are ranked by BIC, from lowest (best model) to highest (worst model). Note that AIC and BIC may differ because BIC imposes a greater penalty on more complex models.

Exhibit 8: All 31 Models of Excess Portfolio Returns Regressed on Up to Five Factors



Note: Each model is designated by its factors. For example, "1 2 4" is the model incorporating Factors 1, 2, and 4 only.

The best model of all 31 models is the model with just Factors 1 and 4. This model has the lowest BIC and AIC and the highest adjusted R^2. Note the model with the highest R^2 has all five factors and ranks relatively poorly on all the other metrics. This visual demonstrates the consistency between AIC and BIC when comparing models and how these statistics differ from the R^2 and adjusted R^2 in selecting the best model.

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KNOWLEDGE CHECK

You are a junior analyst tasked with determining important signals of higher ROA for companies. At today's investment meeting, the PM suggested three possible reasons: higher capital investment, higher advertising costs, and higher R&D spending. You cover the manufacturing sector, and you want to determine which factors, if any, signal improved ROA. You estimate the following three models for ROA for a sample of 26 diversified manufacturers:

Model 1: $\text{ROA}_i = b_0 + b_{CAPEX}\text{CAPEX}_i + \varepsilon_i$;

Model 2: $\text{ROA}_i = b_0 + b_{CAPEX}\text{CAPEX}_i + b_{ADV}\text{ADV}_i + \varepsilon_i$;

Model 3: $\text{ROA}_i = b_0 + b_{CAPEX}\text{CAPEX}_i + b_{ADV}\text{ADV}_i + b_{R\&D}\text{R\&D}_i + \varepsilon_i$

Where:

- CAPEX is capital expenditures scaled by beginning-year PPE;
- ADV is the prior year's ratio of advertising expenditures to revenues; and
- R&D is the prior period's R&D expenditures divided by revenues.

Partial ANOVA Results for Model 1

	Df	*SS*	*MS*	*F*	Significance *F*
Regression	1	410.3606	410.3606	175.8896	0.0000
Residual	24	55.9934	2.3331		
Total	25	466.3540			
Log-likelihood	–46.842				

Partial ANOVA Results for Model 2

	Df	*SS*	*MS*	*F*	Significance *F*
Regression	2	410.6278	205.3139	84.7396	0.0000
Residual	23	55.7263	2.4229		
Total	25	466.3540			
Log-likelihood	–46.795				

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ANOVA Results for Model 3

Source	Degrees of freedom	Sum of squares	Mean squares	*F*-statistic	Significance of *F*-statistic
Regression	3	410.9591	136.9864	54.4039	0.0000
Residual	22	55.3949	2.5180		
Total	25	466.3540			
Log-likelihood	–46.716				

					95% confidence interval	
	Coefficient	Standard error	*t*-Statistic	*P*-value	Lower bound	Upper bound
Intercept	4.7022	1.2349	3.8078	0.0010	2.1412	7.2633
CAPEX	1.2302	0.1056	11.6490	0.0000	1.0112	1.4492
ADV	–0.0371	0.1062	–0.3490	0.7304	–0.2572	0.1831
R&D	0.1029	0.2837	0.3628	0.7203	–0.4854	0.6913

Before the next investment meeting, the PM asks you to do the following:

1. Demonstrate the relationship between R^2 and adjusted R^2 for Model 3.

Solution

For Model 3, the R^2 is 0.8812 and the adjusted R^2 is 0.8650, as follows:

$$R^2 = \frac{410.9591}{466.3540} = 0.8812.$$

$$\text{Adjusted } R^2 = 1 - \left[\frac{25}{22}(1 - 0.8812)\right] = 0.8650.$$

2. Determine whether Model 3 explains ROA using a 1% significance level.

Solution

The test of the full (unrestricted) Model 3 is as follows:

Step 1	State the hypotheses.	H_0: $b_{CAPEX} = b_{ADV} = b_{R\&D} = 0$. H_a: At least one $b_j \neq 0$.
Step 2	Identify the appropriate test statistic.	$F = \frac{\text{MSR}}{\text{MSE}}$, with 3 and 22 degrees of freedom.
Step 3	Specify the level of significance.	α = 1% (one-tail, right side).
Step 4	State the decision rule.	Critical *F*-value = 4.8166. Reject the null hypothesis if the calculated *F*-statistic exceeds 4.8166.

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Step 5	Calculate the test statistic.	F = 54.4039, as given in the regression output. (*Note small difference vs. MSR/MSE from rounding.)*
Step 6	Make a decision.	Reject the null hypothesis because the calculated F-statistic exceeds the critical F-value. There is sufficient evidence that at least one slope coefficient is different from zero.

3. Determine whether each slope coefficient in Model 3 is different from zero at the 1% significance level.

Solution

The test of whether each slope coefficient in Model 3 is different from zero is as follows:

Step 1	State the hypotheses.	H_0: $b_j = 0$ vs. H_a: $b_j \neq 0$.
Step 2	Identify the appropriate test statistic.	$t = \frac{\hat{b}_j - B_j}{s_{\hat{b}j}}$, with 26 − 3 − 1 = 22 degrees of freedom.
Step 3	Specify the level of significance.	α = 1% (two-tail).
Step 4	State the decision rule.	Critical t-values = ± 2.8188. Reject the null hypothesis if the calculated t-statistic is greater than 2.8188 or less than −2.8188.
Step 5	Calculate the test statistic.	CAPEX: $t = \frac{1.2302 - 0}{0.1056} = 11.649$. ADV: $t = \frac{-0.0371 - 0}{0.1062} = 0.3493$. R&D: $t = \frac{0.1029 - 0}{0.2837} = 0.3627$. *Note calculated* t*-statistics may differ from output due to rounding.*
Step 6	Make a decision.	Reject the null hypothesis for CAPEX because the calculated t-statistic exceeds the critical t-value. There is sufficient evidence to indicate the slope coefficient for CAPEX is different from zero. Fail to reject the null hypothesis for both the ADV and R&D variables because the calculated t-statistics are within the bounds of the critical values. There is not sufficient evidence to indicate that the slope coefficients of ADV or R&D are different from zero.

4. Determine whether ADV and R&D together contribute to the explanation of ROA in Model 3 at a 1% significance level using the joint F-test.

Solution

The joint test of whether ADV and R&D together contribute to the explanation of ROA in Model 3 is as follows:

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Step 1	State the hypotheses.	H_0: $b_{ADV} = b_{R\&D} = 0$ vs. H_a: At least one $b_j \neq 0$.
Step 2	Identify the appropriate test statistic.	$F = \frac{(\text{Sum of squares error restricted model} - \text{Sum of squares error unrestricted model})/q}{\text{Sum of squares error unrestricted model}/(n-k-1)}$, with 2 and 22 degrees of freedom
Step 3	Specify the level of significance.	α = 1% (one-tail, right side).
Step 4	State the decision rule.	Critical *F*-value = 5.7190. Reject the null hypothesis if the calculated *F*-statistic exceeds 5.7190.
Step 5	Calculate the test statistic.	$F = \frac{(55.9934 - 55.3949)/2}{55.3949/22} = 0.1188$.
Step 6	Make a decision.	Fail to reject the null hypothesis because the calculated *F*-statistic does not exceed the critical *F*-value. There is not sufficient evidence to indicate that at least one of the two slope coefficients is different from zero. This evidence suggests that Model 1, using only CAPEX, is the better model compared to Model 3.

5. State and justify your conclusion based on these tests of hypotheses.

Solution

The conclusion is the variation of ROA in this sample is explained by CAPEX. The variables of ADV and R&D by themselves or together do not explain the variation in ROA. The justification for this conclusion is as follows:

- Adjusted R^2 is higher for Model 1 (CAPEX) versus Model 3 (all three variables).
- ADV and R&D are each insignificant based on *t*-tests of Model 3.
- ADV and R&D together are insignificant based on the joint *F*-test using Model 1 (restricted) and Model 3 (unrestricted).

6. Consider the following statistics on all possible ROA models:

Variable(s) in Model	R^2	Adjusted R^2	AIC	BIC
CAPEX	0.8803	0.8753	23.8988	26.5226
R&D	0.0866	0.0486	76.7297	73.9100
ADV	0.0619	0.0228	77.4233	74.5737
CAPEX, R&D	0.8809	0.8706	25.7594	28.7575
CAPEX, ADV	0.8807	0.8703	25.8044	28.7917
ADV, R&D	0.1473	0.0731	76.9431	72.5325
CAPEX, ADV, R&D	0.8814	0.8653	27.6481	31.0365

Recommend the best model from the set and justify your choice.

Solution

The simple regression model with CAPEX alone is the best model among those presented. This model has the highest adjusted R^2 and the lowest AIC and BIC. These model-fit statistics confirm that the model with CAPEX alone, which is the most parsimonious, is the best-fitting model.

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FORECASTING USING MULTIPLE REGRESSION

4

☐ calculate and interpret a predicted value for the dependent variable, given the estimated regression model and assumed values for the independent variable

The process for predicting the value of the dependent variable using an estimated multiple regression model is similar to that in simple regression, but with more items to sum, as shown in the following formula:

$$\hat{Y}_f = \hat{b}_0 + \hat{b}_1 X_{1f} + \hat{b}_2 X_{2f} + \ldots + \hat{b}_k X_{kf} = \hat{b}_0 + \sum_{j=1}^{k} \hat{b}_j X_{jf} \quad (6)$$

As the formula indicates, to determine the predicted (forecasted) value of the dependent variable $\left(\hat{Y}_f\right)$, the analyst must sum, for each independent variable, the estimated slope coefficient multiplied by the assumed value for that variable $\left(\hat{b}_j X_f\right)$ and then add the estimated intercept coefficient $\left(\hat{b}_0\right)$ times the assumed value for the intercept of 1.

We can use the regression results from Exhibit 1 of portfolio returns on the five factors to predict the portfolio return using the following model:

$$Y_i = b_0 + b_1 \text{Factor}_{1i} + b_2 \text{Factor}_{2i} + b_3 \text{Factor}_{3i} + b_4 \text{Factor}_{4i} + b_5 \text{Factor}_{5i} + \varepsilon_i,$$

which now uses assumed values of the factors. This model becomes

$$\hat{Y}_f = -2.1876 + 1.5992\ \text{Factor}_1 + 0.1923\ \text{Factor}_2 - 0.7126\ \text{Factor}_3 + 3.3376\ \text{Factor}_4 - 2.6832\ \text{Factor}_5,$$

and the assumed values of the five factors are as follows:

Factor	1	2	3	4	5
Assumed Value	0.110	0.040	0.080	–0.010	0.001

Exhibit 9 shows the calculation of the predicted value of the portfolio return, –2.0971%.

Exhibit 9: Predicting Portfolio Returns Using the Five-Factor Model

	(1)	(2)	(1) × (2)
	Assumed value	***Estimated coefficient***	***Product***
Intercept	1.000	–2.1876	–2.1876
Factor 1	0.110	1.5992	0.1759
Factor 2	0.040	0.1923	0.0077
Factor 3	0.080	–0.7126	–0.0570
Factor 4	–0.010	3.3376	–0.0334
Factor 5	0.001	–2.6832	–0.0027
			–2.0971

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There are cautions regarding predicting with a multiple regression model:

- If a regression model is estimated using all five independent variables, for example, any prediction of the dependent variable must also include all five variables—even the ones that are not statistically significant. This is because correlations between these variables were used in estimating the slope coefficients.
- For any prediction of the dependent variable, we must also include the intercept term.

As with simple linear regression forecasts, we are often interested in the level of uncertainty around the forecast of the dependent variable in terms of the standard error of the forecast. In any regression estimation, there are residuals because not all observations lie on the estimated line. This is basic uncertainty in the model—the *model error*—which is the stochastic part of the model that involves the regression residual, ε_i.

When the independent variables themselves are forecasts and thus out-of-sample predictions, there is an added source of error arising from errors associated with forecasting the independent variables. In such cases, the forecast error for the dependent variable is dependent on how well the independent variables (X_{1f}, X_{2f}, …, X_{kf}) were forecasted, hence introducing *sampling error*. The combined effect of the model error and the sampling error results in a standard error of the forecast for $\hat{Y}_f$ that is larger than the standard error of the regression. This larger forecast error results in a prediction interval for the dependent variable that is wider than the within-sample error.

Although the calculation of the forecast interval for multiple regression is overly detailed for our purposes, we can use software to produce this interval and the corresponding standard error of the forecast. For the five-factor portfolio return model, the standard error of the forecast is 1.1466, the upper 95% confidence bound is 0.2119, and the lower 95% confidence bound is –4.4098, compared with the point estimate, noted above, of –2.0971.

KNOWLEDGE CHECK

1. Your research report includes the following regression equation:

   $$\text{ROA} = 4.7022 + 1.2302\text{CAPEX} - 0.0371\text{ADV} + 0.1029\text{R\&D}.$$

   An institutional salesperson at your firm asks you to determine the predicted ROA for a company with assumed values for the three independent variables of CAPEX = 5%, ADV = 4%, and R&D = 3%.

   Solution

   The predicted ROA is 11.0135%, calculated as follows:

   $$\widehat{\text{ROA}}_f = 4.7022 + 1.2302(5) - 0.0371(4) + 0.1029(3) = 11.0135.$$

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PRACTICE PROBLEMS

The following information relates to questions 1-5

You are a junior analyst at an asset management firm. Your supervisor asks you to analyze the return drivers for one of the firm's portfolios. She asks you to construct a regression model of the portfolio's monthly excess returns (RET) against three factors: the market excess return (MRKT), a value factor (HML), and the monthly percentage change in a volatility index (VIX). You collect the data and run the regression. After completing the first regression (Model 1), you review the ANOVA results with your supervisor.

Then, she asks you to create two more models by adding two more explanatory variables: a size factor (SMB) and a momentum factor (MOM). Your three models are as follows:

Model 1: $\text{RET}_i = b_0 + b_{MRKT}\text{MRKT}_i + b_{HML}\text{HML}_i + b_{VIX}\text{VIX}_i + \varepsilon_i$.

Model 2: $\text{RET}_i = b_0 + b_{MRKT}\text{MRKT}_i + b_{HML}\text{HML}_i + b_{VIX}\text{VIX}_i + b_{SMB}\text{SMB}_i + \varepsilon_i$.

Model 3: $\text{RET}_i = b_0 + b_{MRKT}\text{MRKT}_i + b_{HML}\text{HML}_i + b_{VIX}\text{VIX}_i + b_{SMB}\text{SMB}_i + b_{MOM}\text{MOM}_i + \varepsilon_i$.

The regression statistics and ANOVA results for the three models are shown in Exhibit 1, Exhibit 2, and Exhibit 3.

Exhibit 1: ANOVA Table for Model 1

$\text{RET}_i = b_0 + b_{MRKT}\text{MRKT}_i + b_{HML}\text{HML}_i + b_{VIX}\text{VIX}_i + \varepsilon_i$

Regression Statistics			*Coefficient*	*Std. Error*	*t-Stat.*	*P-Value*
Multiple *R*	0.907	Intercept	–0.999	0.414	–2.411	0.018
R-Squared	0.823	MRKT	1.817	0.124	14.683	0.000
Adjusted *R*-Sq.	0.817	HML	0.489	0.118	4.133	0.000
Standard Error	3.438	VIX	0.037	0.018	2.122	0.037
Observations	96.000					

ANOVA

	Df	*SS*	*MS*	*F*	*Significance F*
Regression	3	5058.430	1686.143	142.628	0.000
Residual	92	1087.618	11.822		
Total	95	6146.048			

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Exhibit 2: ANOVA Table for Model 2

$RET_i = b_0 + b_{MRKT}MRKT_i + b_{HML}HML_i + b_{VIX}VIX_i + b_{SMB}SMB_i + \varepsilon_i$

Regression Statistics			*Coefficient*	*Std. Error*	*t-Stat.*	*P-Value*
Multiple *R*	0.923	Intercept	–0.820	0.383	–2.139	0.035
R-Squared	0.852	MRKT	1.649	0.121	13.683	0.000
Adjusted *R*-Sq.	0.846	HML	0.434	0.109	3.970	0.000
Standard Error	3.161	VIX	0.025	0.016	1.516	0.133
Observations	96.000	SMB	0.563	0.133	4.223	0.000

ANOVA

	Df	*SS*	*MS*	*F*	*Significance F*
Regression	4	5236.635	1309.159	131.000	0.000
Residual	91	909.413	9.994		
Total	95	6146.048			

Exhibit 3: ANOVA Table for Model 3

$RET_i = b_0 + b_{MRKT}MRKT_i + b_{HML}HML_i + b_{VIX}VIX_i + b_{SMB}SMB_i + b_{MOM}MOM_i + \varepsilon_i$

Regression Statistics			*Coefficient*	*Std. Error*	*t-Stat.*	*P-Value*
Multiple *R*	0.923	Intercept	–0.823	0.385	–2.136	0.035
R-Squared	0.852	MRKT	1.719	0.280	6.130	0.000
Adjusted *R*-Sq.	0.844	HML	0.412	0.138	2.989	0.004
Standard Error	3.177	VIX	0.026	0.017	1.532	0.129
Observations	96.000	SMB	0.553	0.139	3.987	0.000
		MOM	–0.067	0.242	–0.276	0.783

ANOVA

	Df	*SS*	*MS*	*F*	*Significance F*
Regression	5	5237.402	1047.480	103.751	0.000
Residual	90	908.647	10.096		
Total	95	6146.048			

Your supervisor asks for your assessment of the model that provides the best fit as well as the model that is best for predicting values of the monthly portfolio return. So, you calculate Akaike's information criterion (AIC) and Schwarz's Bayesian information criterion (BIC) for all three models, as shown in Exhibit 4.

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Exhibit 4: Goodness-of-Fit Measures

	AIC	BIC
Model 1	241.03	251.29
Model 2	225.85	238.67
Model 3	227.77	243.16

1. Determine which one of the following reasons for the change in adjusted R^2 from Model 2 to Model 3 is *most likely* to be correct.
   - **A.** Adjusted R^2 decreases since adding MOM does not improve the overall explanatory power of Model 3.
   - **B.** Adjusted R^2 increases since adding SMB improves the overall explanatory power of Model 2.
   - **C.** Adjusted R^2 decreases since adding MOM improves the overall explanatory power of Model 3.

2. Identify the model that provides the best fit.
   - **A.** Model 1
   - **B.** Model 2
   - **C.** Model 3

3. Identify the model that should be used for prediction purposes.
   - **A.** Model 1
   - **B.** Model 2
   - **C.** Model 3

4. Calculate the predicted RET for Model 3 given the assumed factor values: MRKT = 3, HML = –2, VIX = –5, SMB = 1, MOM = 3.
   - **A.** 3.732
   - **B.** 3.992
   - **C.** 4.555

5. Calculate the joint *F*-statistic and determine whether SMB and MOM together contribute to explaining RET in Model 3 at a 1% significance level (use a critical value of 4.849).
   - **A.** 2.216, so SMB and MOM together do not contribute to explaining RET
   - **B.** 8.863, so SMB and MOM together do contribute to explaining RET
   - **C.** 9.454, so SMB and MOM together do contribute to explaining RET

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SOLUTIONS

1. A is correct. Adjusted R^2 in Model 3 decreases to 0.844 from 0.846 in Model 2. Model 3 includes all independent variables from Model 2, while adding MOM. Adding variables to a regression model always either increases R^2 or causes it to stay the same. But adjusted R^2 only increases if the new variable meets a threshold of significance, $|t\text{-statistic}| > 1$. MOM does not meet this threshold, indicating it does not improve the overall explanatory power of Model 3.

2. B is correct. BIC is the preferred measure for determining which model provides the best fit, and a lower BIC is better. Since Model 2 has the lowest BIC value, it provides the best fit among the three models.

3. B is correct. AIC is the preferred measure for determining the model that is best used for prediction purposes. As with BIC, a lower AIC is better. Model 2 also has the lowest AIC value among the three models; thus, it should be used for prediction purposes.

4. A is correct. The regression equation for Model 3 is

   RET = –0.823 + 1.719MRKT + 0.412HML + 0.026VIX + 0.553SMB – 0.067MOM.

   Using the assumed values for the independent variables, we have

   RET = –0.823 + (1.719)(3) + (0.412)(–2) + (0.026)(–5) + (0.553)(1) – (0.067)(3) = 3.732.

5. B is correct. To determine whether SMB and MOM together contribute to the explanation of RET, at least one of the coefficients must be non-zero. So, H_0: $b_{SMB} = b_{MOM} = 0$ and H_a: $b_{SMB} \neq 0$ and/or $b_{MOM} \neq 0$.
   We use the *F*-statistic, where

   $$F = \frac{(\text{SSE of restricted model} - \text{SSE of unrestricted model})/q}{\text{SSE of unrestricted model}/(n-k-1)},$$

   with $q = 2$ and $n - k - 1 = 90$ degrees of freedom. The test is one-tailed, right side, with $\alpha = 1\%$, so the critical *F*-value is 4.849.
   Model 1 does not include SMB and MOM, so it is the restricted model. Model 3 includes all of the variables of Model 1 as well as SMB and MOM, so it is the unrestricted model.
   Using data in Exhibit 1 and Exhibit 3, the joint *F*-statistic is calculated as

   $$F = \frac{(1087.618 - 908.647)/2}{908.647/90} = \frac{89.485}{10.096} = 8.863.$$

   Since 8.863 > 4.849, we reject H_0. Thus, SMB and MOM together do contribute to the explanation of RET in Model 3 at a 1% significance level.

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LEARNING MODULE

3

Model Misspecification

LEARNING OUTCOMES

Mastery	*The candidate should be able to:*
☐	describe how model misspecification affects the results of a regression analysis and how to avoid common forms of misspecification
☐	explain the types of heteroskedasticity and how it affects statistical inference
☐	explain serial correlation and how it affects statistical inference
☐	explain multicollinearity and how it affects regression analysis

INTRODUCTION

1

LEARNING MODULE OVERVIEW

- Principles for proper regression model specification include economic reasoning behind variable choices, parsimony, good out-of-sample performance, appropriate model functional form, and no violations of regression assumptions.
- Failures in regression functional form are typically due to omitted variables, inappropriate form of variables, inappropriate variable scaling, and inappropriate data pooling; these may lead to the violations of regression assumptions.
- Heteroskedasticity occurs when the variance of regression errors differs across observations. Unconditional heteroskedasticity is when the error variance is not correlated with the independent variables, whereas conditional heteroskedasticity exists when the error variance is correlated with the values of the independent variables.
- Unconditional heteroskedasticity creates no major problems for statistical inference, but conditional heteroskedasticity is problematic because it results in underestimation of the regression coefficients' standard errors, so *t*-statistics are inflated and Type I errors are more likely.

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- Conditional heteroskedasticity can be detected using the Breusch–Pagan (BP) test, and the bias it creates in the regression model can be corrected by computing robust standard errors.
- Serial correlation (or autocorrelation) occurs when regression errors are correlated across observations and may be a serious problem in time-series regressions. Serial correlation can lead to inconsistent coefficient estimates, and it underestimates standard errors, so *t*-statistics are inflated (as with conditional heteroskedasticity).
- The Breusch–Godfrey (BG) test is a robust method for detecting serial correlation. The BG test uses residuals from the original regression as the dependent variable run against initial regressors plus lagged residuals, and H_0 is the coefficients of the lagged residuals are zero.
- The biased estimates of standard errors caused by serial correlation can be corrected using robust standard errors, which also correct for conditional heteroskedasticity.
- Multicollinearity occurs with high pairwise correlations between independent variables or if three or more independent variables form approximate linear combinations that are highly correlated. Multicollinearity results in inflated standard errors and reduced *t*-statistics.
- The variance inflation factor (VIF) is a measure for quantifying multicollinearity. If VIF_j is 1 for X_j, then there is no correlation between X_j and the other regressors. $VIF_j > 5$ warrants further investigation, and $VIF_j > 10$ indicates serious multicollinearity requiring correction.
- Solutions to multicollinearity include dropping one or more of the regression variables, using a different proxy for one of the variables, or increasing the sample size.

2 MODEL SPECIFICATION ERRORS

☐ describe how model misspecification affects the results of a regression analysis and how to avoid common forms of misspecification

Model specification refers to the set of variables included in the regression and the regression equation's functional form. Here we provide broad guidelines for correctly specifying a regression and then describe common types of model misspecification.

The principles for good regression model specification are presented concisely in Exhibit 1.

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Exhibit 1: Principles for Proper Regression Model Specification

Principle	Explanation
Model should be grounded in economic reasoning.	Provide economic reasoning behind choice of variables.
Model should be parsimonious.	Each variable included in regression should play an essential role.
Model should perform well out of sample.	Model may explain only the specific dataset on which it was trained, meaning it is overfit.
Model functional form should be appropriate.	If a nonlinear relationship between regressors is expected, model should incorporate the appropriate nonlinear terms.
Model should satisfy regression assumptions.	If heteroskedasticity, serial correlation, or multicollinearity are detected, revise regression variables and/or functional form.

We now cover model specification errors. Understanding them will lead to better model development and more informed use of investment research.

MISSPECIFIED FUNCTIONAL FORM

3

When estimating a regression, we assume it has the correct functional form, an assumption that can fail in different ways, as shown in Exhibit 2.

Exhibit 2: Failures in Regression Functional Form

Failures in Regression Functional Form	Explanation	Consequence
Omitted variables	One or more important variables are omitted from the regression.	May lead to heteroskedasticity or serial correlation
Inappropriate form of variables	Ignoring a nonlinear relationship between the dependent and independent variable	May lead to heteroskedasticity
Inappropriate variable scaling	One or more regression variables may need to be transformed before estimating the regression.	May lead to heteroskedasticity or multicollinearity
Inappropriate data pooling	Regression model pools data from different samples that should not be pooled.	May lead to heteroskedasticity or serial correlation

Omitted Variables

First, consider **omitted variable bias**, the bias resulting from the omission of an important independent variable from a regression. If the true regression model is

$$Y_i = b_0 + b_1 X_{1i} + b_2 X_{2i} + \varepsilon_i$$

but we estimate the model as

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$$Y_i = b_0 + b_1 X_{1i} + \varepsilon_i,$$

the latter model is misspecified because X_2 is omitted.

If the omitted variable is uncorrelated with X_1, the residual will be $b_2 X_{2i} + \varepsilon_i$. Therefore, the residual in the misspecified regression will not have an expected value of zero nor will it be independent and identically distributed, depending on the behavior of X_2. This means that the estimate of the intercept will be biased, although in this instance, the coefficient for X_1 will still be estimated correctly.

If instead the omitted variable (X_2) is correlated with the remaining variable (X_1), the error term in the model will be correlated with X_1 and the estimated values of the regression coefficients in the latter model will be biased and inconsistent. The estimated coefficient on X_1, the intercept, and the residuals will be incorrect. The estimates of the coefficients' standard errors will also be inconsistent, so these cannot be used for conducting statistical tests.

Inappropriate Form of Variables

Another misspecification error in regression models is using the wrong form of the data when a transformed version is appropriate. For example, an analyst may fail to account for nonlinearity in the relationship between the dependent variable and one or more independent variables by specifying a linear relation. When specifying a regression model, we should consider whether economic theory suggests a nonlinear relation. We can often confirm nonlinearity by plotting the data. For example, if the relation between variables becomes linear when one or more of the variables is represented as a proportional change in the variable, then we can correct the misspecification by taking the natural logarithm of the variable we want to represent as a proportional change.

Inappropriate Scaling of Variables

The use of unscaled data in regressions when scaled data are more appropriate may cause model misspecification. Often, analysts must decide whether to scale variables before they compare data across companies. For example, analysts often compare companies using common-size financial statements. Common-size statements make comparability across companies much easier, allowing the analyst to quickly compare trends in profitability, leverage, efficiency, and so on, for a group of companies.

Inappropriate Pooling of Data

Finally, another common misspecification error is pooling data from samples that should not be pooled. Inappropriate pooling data may occur when the sample spans structural breaks in the behavior of the data. This might arise from a change in government regulation or a regime change from a low-volatility period to a high-volatility period. In a scatterplot, such data would appear in discrete, widely separated clusters with little or no correlation, because the means of the data for each cluster would be very different. When available data results from discernible subsamples, the analyst should estimate the model using the subsample most representative of conditions during the forecasting period.

KNOWLEDGE CHECK

You are a junior analyst at a firm specializing in precious metals funds. The firm's outlook is for increasing stock market volatility over the next six months, so the research director tasks you with modeling the relation between gold returns and changes in stock market volatility. You collect 32 months of

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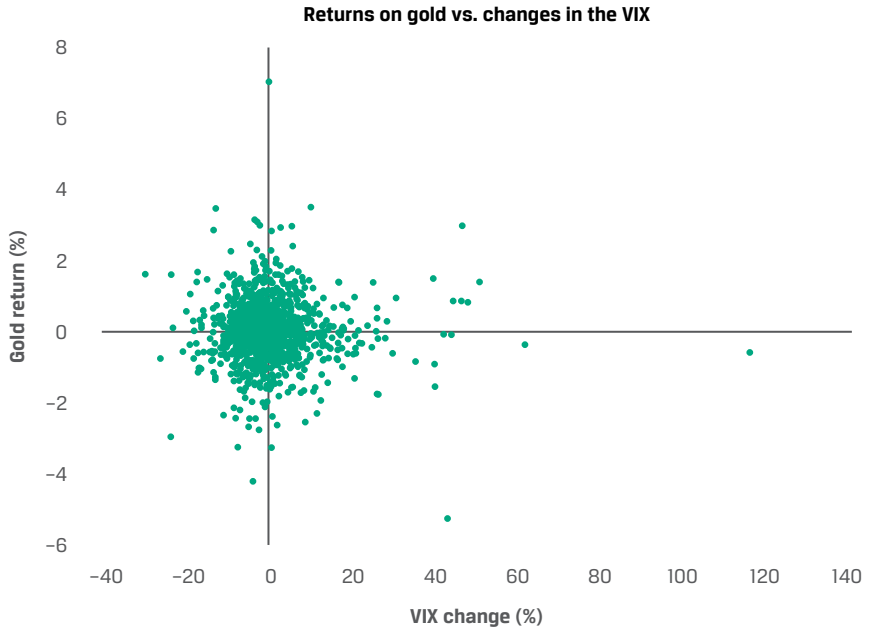
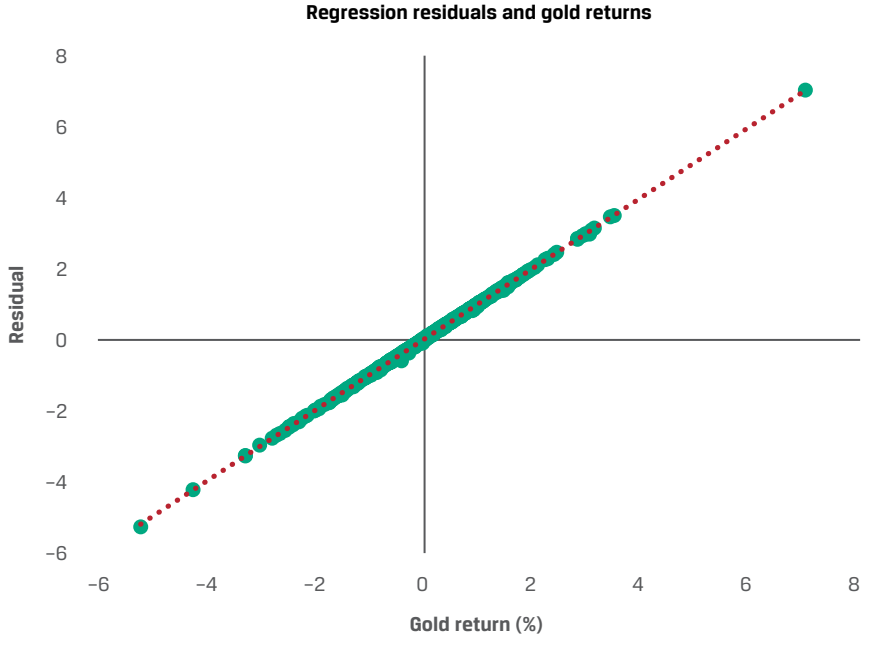
returns to gold (GOLD) and changes in the CBOE VIX Index (VIX), estimate a model with GOLD as the dependent variable and VIX as the independent variable, and create the following two scatterplots.





1. Discuss whether these scatterplots provide evidence of any violations of regression assumptions.

Solution

The residuals display a strong positive relationship (not noise) versus GOLD, which is a concern. This suggests the residuals are not normally distributed, are correlated, have a non-zero expected value, and are heteroskedastic. The conclusion, apparent in the lack of linear relation between gold returns and changes in the VIX, is that VIX does not capture the entire explanation of the variation in GOLD.

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2. Describe how to address any violations that may be indicated.

Solution

The model should be modified to include additional features—such as market returns, economic drivers such as expected inflation, and even an indicator variable to classify periods of high versus low geopolitical risk—that may better explain the variation in gold returns.

Next, you are assigned to study a factor model for explaining the excess returns to precious metals Portfolio A (PORTA), estimated as

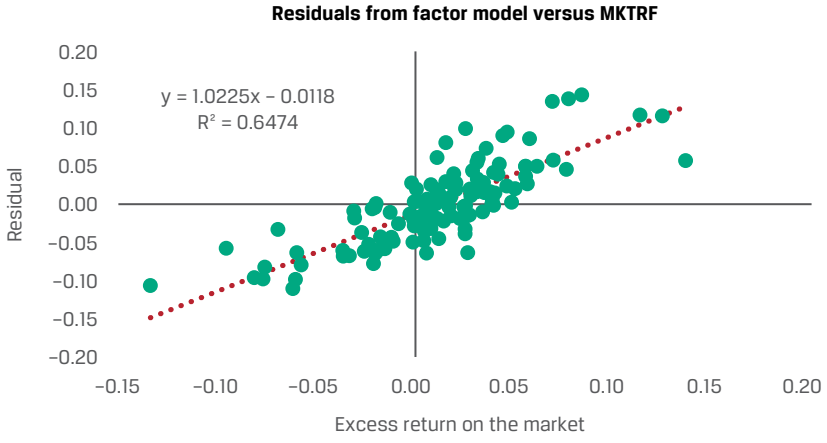
$\text{PORTA}_i = b_0 + b_1X_{1i} + b_2X_{2i} + \varepsilon_i$,

with the following results:

	Coefficient	*Standard Error*	t-*Stat.*	P-*Value*
Intercept	0.013	0.005	2.671	0.009
X_1, Factor 1	1.896	0.208	9.111	0.000
X_2, Factor 2	0.170	0.186	0.916	0.361

Also, adjusted R^2 is 43%, and the general F-test indicates rejection of the null hypothesis that the coefficients on X_1 and X_2 are equal to zero.

You are asked if it is appropriate to include excess stock market return (MKTRF) as an additional explanatory variable, so you examine the residuals from the estimated model and their relationship with MKTRF using the following scatterplot:



3. Identify the type of model misspecification suggested by the scatterplot.

Solution

The model has two variables (X_1 and X_2), but the residuals from this model have a clear relationship with MKTRF. Therefore, the type of misspecification error is the omitted variable bias.

4. Describe how the misspecification affects the estimated model.

Solution

The estimated coefficients for X_1 and X_2 are biased and inconsistent, as are their standard errors, so the model cannot be relied on for making statistical inferences.

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5. Recommend a correction for the misspecification.

Solution

The recommended correction for this misspecification error is to include the omitted variable, MKTRF, and re-estimate the revised model.

VIOLATIONS OF REGRESSION ASSUMPTIONS: HETEROSKEDASTICITY

4

☐ explain the types of heteroskedasticity and how it affects statistical inference

An important assumption underlying linear regression is that the variance of errors is constant across observations (errors are homoskedastic). Residuals in financial model estimations, however, are often **heteroskedastic**, meaning the variance of the residuals differs across observations. Heteroskedasticity may arise from model misspecification, including omitted variables, incorrect functional form, and incorrect data transformations, as well as from extreme values of independent variables.

The difference between homoskedastic and heteroskedastic errors can be seen in Exhibit 3. Panel A shows the values of the dependent and independent variables and a fitted regression line for a model with homoskedastic errors; there is no apparent relationship between the regression residuals and the value of the independent variable. Panel B shows the relationship when there are heteroskedastic errors. There is a systematic relationship between the value of the independent variable and the regression residuals: The residuals, which are the distance from the line, are larger for larger values of the independent variable.

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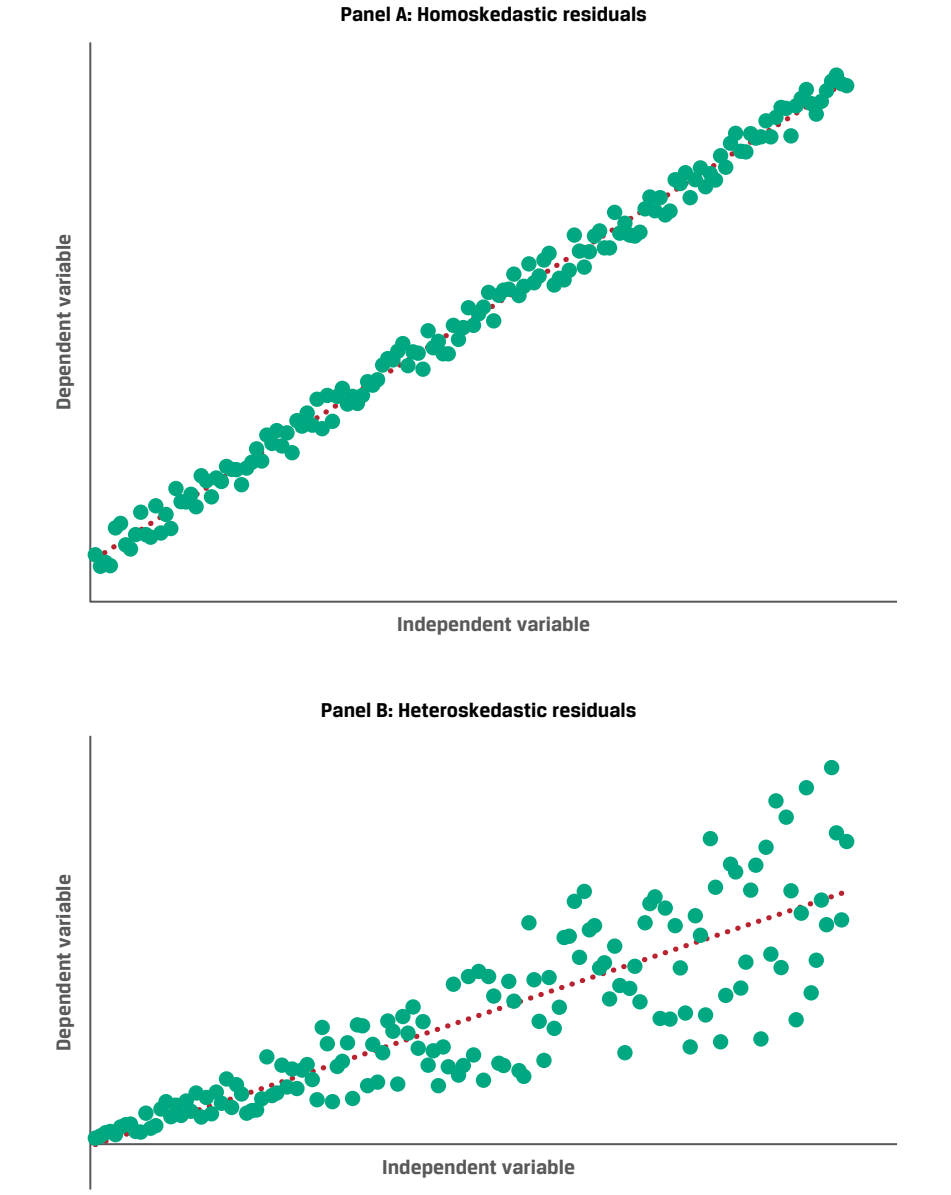
Exhibit 3: Homoskedastic and Heteroskedastic Residuals



The Consequences of Heteroskedasticity

There are two broad types of heteroskedasticity: unconditional and conditional. **Unconditional heteroskedasticity** occurs when the error variance is not correlated with the regression's independent variables. Although it violates a linear regression assumption, this form of heteroskedasticity creates no major problems for statistical inference.

Conditional heteroskedasticity is more problematic for statistical inference—when the error variance is correlated with (conditional on) the values of the independent variables. This type of heteroskedasticity may lead to mistakes in statistical inference. When errors are conditional heteroskedastic, the *F*-test for the overall regression significance is unreliable because the MSE becomes a biased estimator of the true

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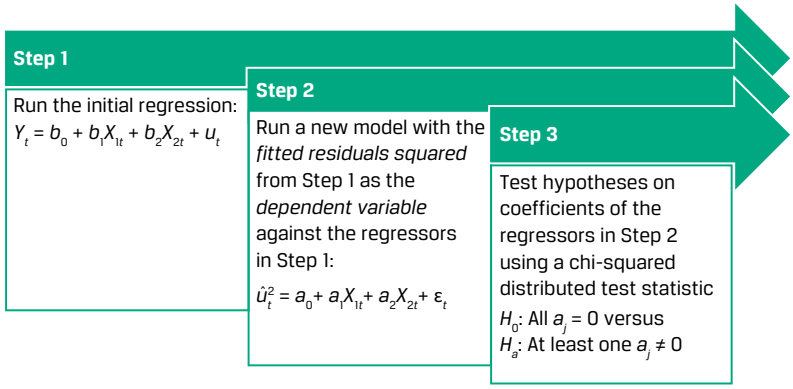
population variance. Moreover, *t*-tests of individual regression coefficients are unreliable because heteroskedasticity introduces bias into estimators of the standard error of regression coefficients. Thus, in regressions with financial data, the most likely impacts of conditional heteroskedasticity are that standard errors will be underestimated, so *t*-statistics will be inflated. If there is conditional heteroskedasticity in the estimated model, we tend to find significant relationships where none actually exist and commit more Type I errors (rejecting the null hypothesis when it is actually true).

Testing for Conditional Heteroskedasticity

The **Breusch–Pagan (BP) test** is widely used in financial analysis to diagnose potential conditional heteroskedasticity and is best understood via the three-step process shown in Exhibit 4. Fortunately, many statistical software packages easily test for and correct conditional heteroskedasticity.

Exhibit 4: Breusch–Pagan Test for Conditional Heteroskedasticity



If conditional heteroskedasticity is present in the initial regression, the independent variables will explain a significant portion of the variation in the squared residuals in Step 2. This is because each observation's squared residual is correlated with the independent variables if the independent variables affect the variance of the errors.

The BP test statistic is approximately chi-square distributed with k degrees of freedom, where k is the number of independent variables in Step 1:

$$X^2_{BP,k} = nR^2, \tag{1}$$

and here R^2 is from Step 2. The null hypothesis is that there is no conditional heteroskedasticity; the regression's squared residuals are uncorrelated with the independent variables. The alternative is that there is correlation with at least one independent variable. This is a one-tail, right-side test.

The BP test is illustrated in Exhibit 5. Here, an analyst uses 10 years of monthly data to assess exposures of Stock XYZ's excess returns (STOCK_RETRF) using the Fama–French three-factor model; the regressors are excess market return (MKTRF), the size factor (SMB), and the value factor (HML). The regression output is in Panel A.

The regression yields significant estimated MKTRF (1.2414) and SMB (1.0953) exposures. To validate the results, the analyst can conduct a residual analysis by plotting regression residuals against each of the three factors to look for clues to violations of

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linear regression assumptions, including conditional heteroskedasticity. However, the BP test (results shown in Panel B) is more rigorous. In this case, the null hypothesis is rejected at the 1% level of significance, indicating conditional heteroskedastic residuals.

Exhibit 5: Testing for Conditional Heteroskedasticity in Fama–French Three-Factor Model

Panel A Explaining XYZ Returns Using Fama–French Three-Factor Model

Regression Statistics

Multiple *R*	0.9375
R-Squared	0.8788
Adjusted *R*-Squared	0.8757
Standard Error	0.0246
Observations	120

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	3	0.5074	0.1691	280.4951	0.0000
Residual	116	0.0699	0.0006		
Total	119	0.5773			

	Coefficient	Standard Error	*t*-Stat.	*P*-Value
Intercept	–0.0026	0.0024	–1.0735	0.2853
MKTRF	1.2414	0.0601	20.6419	0.0000
SMB	1.0953	0.1042	10.5147	0.0000
HML	–0.1065	0.0872	–1.2205	0.2248

Panel B Breusch–Pagan Test for Heteroskedasticity

BP test statistic	13.40264
P-value	0.00038

CODE: BREUSCH–PAGAN TEST FOR HETEROSKEDASTICITY

Using Python

```
from statsmodels.formula.api import ols

import statsmodels.api as sm

model = ols('XYZ_RETRF ~ MKTRF+SMB+HML',data=df).fit()

print(model.summary())

test = sm.stats.diagnostic.het_breuschpagan(model.resid, model.model.
exog)
```

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```
print(test)
```

Using R

```
library(lmtest)
model <- lm('XYZ_RETRF~ MKTRF+SMB+HML',data=df)
print(summary(model))
bptest(model)
```

Correcting for Heteroskedasticity

It is important to note that market efficiency implies that in efficient markets, heteroskedasticity should generally not be observed in financial data. However, if heteroskedasticity is detected, for example, in the form of volatility clustering—where large (small) changes tend to be followed by large (small) changes—then it presents an opportunity to forecast asset returns that should be exploited to generate alpha. So, analysts should not only correct problems in their models due to heteroskedasticity but also understand the underlying processes in their data and capitalize on them.

The easiest method to correct for the effects of conditional heteroskedasticity in linear regression is to compute **robust standard errors**, which adjust the standard errors of the regression's estimated coefficients to account for the heteroskedasticity. Many software packages easily compute robust standard errors, and we recommend using them. Note that robust standard errors are also known as *heteroskedasticity-consistent standard errors* or *White-corrected standard errors*.

Returning to the prior example, where the model's error variance is heteroskedastic, Exhibit 6 shows the results when the regression coefficients' standard errors are corrected for conditional heteroskedasticity. Comparing these standard errors to those in the initial regression (in Exhibit 5), the standard errors for the MKTRF and SMB increase from 0.060 to 0.091 and from 0.104 to 0.111, respectively. Note the regression coefficients did not change; rather, the problematic standard errors are corrected.

Exhibit 6: Explaining XYZ Returns Using Fama–French Three-Factor Model with Standard Errors Corrected for Conditional Heteroskedasticity

	Coefficient	Standard Error	t-Stat.	P-Value
(Intercept)	–0.0026	0.0021	–1.2347	0.2194
MKTRF	1.2414	0.0910	13.6483	0.0000
SMB	1.0953	0.1111	9.8605	0.0000
HML	–0.1065	0.1015	–1.0488	0.2965

KNOWLEDGE CHECK

The summer intern presents to the investment team three multiple regression models, A, B and C, that use various ESG factors to explain stock returns. The research director then asks you, a junior analyst, to examine them for potential heteroskedasticity. You run scatterplots of the regression residuals against factors X_1, X_2, and X_3, respectively, and Breusch–Pagan tests for each of the models.

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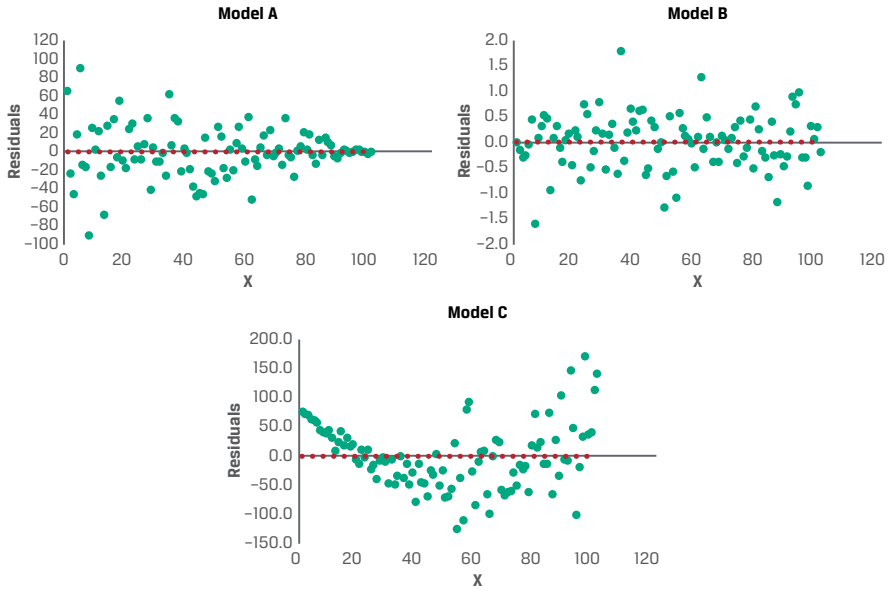

In a follow-up meeting with the research director, she asks you to do the following:

1. Interpret the scatterplot for each regression model, and determine if it suggests heteroskedasticity is a potential problem.

Solution

The Model A scatterplot shows the spread of residuals narrowing dramatically as the value of X_1 increases, suggesting heteroskedasticity is an issue. The Model B scatterplot shows no discernible pattern, just noise, for residuals versus X_2 and thus suggests no heteroskedasticity issue. The Model C scatterplot shows the spread of residuals widening dramatically in a non-linear pattern as the value of X_3 increases, suggesting the presence of both heteroskedasticity and non-linearities.

2. Identify the correct conclusion in the following table for BP tests of the three models using a 5% level of significance.

Model	BP test statistic	*P*-value	Conclusion: Homoskedastic or heteroskedastic residuals
A	7.183	0.0072	
B	0.035	0.8523	
C	29.586	0.0001	

Solution

If the BP test statistic's *P*-value is less than 0.05, then conclude the residuals are heteroskedastic.

	BP test statistic	*P*-value	Conclusion: Homoskedastic or heteroskedastic residuals
A	7.183	0.0072	Heteroskedastic residuals
B	0.035	0.8523	Homoskedastic residuals
C	29.586	0.0001	Heteroskedastic residuals

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3. Given heteroskedastic residuals, describe the expected effect of applying White's correction on coefficient standard errors, calculated *t*-statistics, and corresponding *P*-values for the independent variables.

   Solution

   By applying White's correction for heteroskedasticity, the coefficient standard errors will increase, thereby decreasing estimated *t*-statistics and increasing corresponding *P*-values for the independent variables.

VIOLATIONS OF REGRESSION ASSUMPTIONS: SERIAL CORRELATION

5

☐ explain serial correlation and how it affects statistical inference

A common and serious problem in multiple linear regression is violation of the assumption that regression errors are uncorrelated across observations. When regression errors are correlated across observations, they are serially correlated. **Serial correlation** (or *autocorrelation*) typically arises in time-series regressions. If we have panel data, which is cross-sectional time-series data, serial correlation may also arise. We discuss three aspects of serial correlation: its effect on statistical inference, tests for it, and methods to correct for it.

The Consequences of Serial Correlation

The main problem caused by serial correlation in linear regression is an incorrect estimate of the regression coefficients' standard errors. If none of the regressors is a previous value—a lagged value—of the dependent variable, then the estimated parameters themselves will be consistent and need not be adjusted for the effects of serial correlation. But if one of the independent variables is a lagged value of the dependent variable, serial correlation in the error term causes all parameter estimates to be inconsistent—that is, invalid estimates of the true parameters. These key points are summarized in Exhibit 7.

Exhibit 7: Impact of Serial Correlation on Multiple Regression Model

Independent Variable Is Lagged Value of Dependent Variable	Invalid Coefficient Estimates	Invalid Standard Error Estimates
No	No	Yes
Yes	Yes	Yes

Positive serial correlation is present when a positive residual for one observation increases the chance of a positive residual in a subsequent observation, resulting in a stable pattern of residuals over time. Positive serial correlation also means a negative residual for one observation increases the chance of a negative residual for another observation. Conversely, **negative serial correlation** has the opposite effect, so a positive residual for one observation increases the chance of a negative residual for another observation, and so on. We examine positive serial correlation because it

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is the most common type and assume **first-order serial correlation**, or correlation between adjacent observations. In a time series, this means the sign of the residual tends to persist from one period to the next.

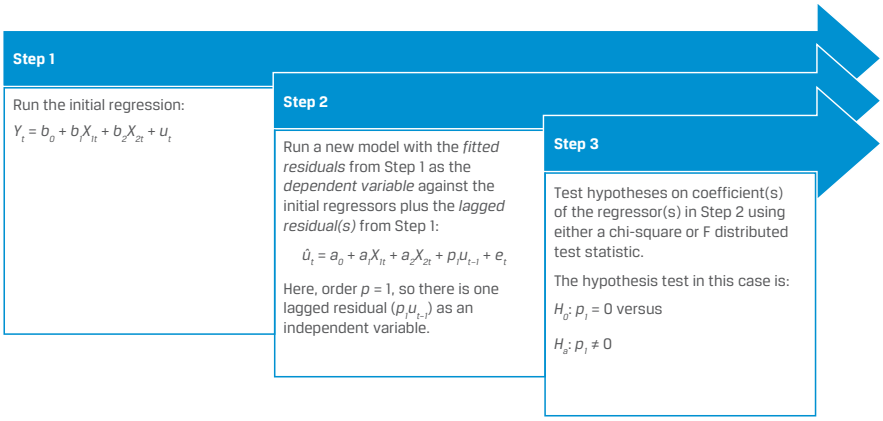
Positive serial correlation does not affect the consistency of regression coefficients, but it does affect statistical tests. First, the *F*-statistic may be inflated because the MSE will tend to underestimate the population error variance. Second, positive serial correlation typically causes standard errors to be underestimated, so *t*-statistics are inflated, which (as with heteroskedasticity) leads to more Type I errors.

Importantly, if a time series exhibits serial correlation, this means that there is some degree of predictability to it. In the case of asset prices, if these prices were to exhibit a pattern, investors would likely discern this pattern and exploit it to capture alpha, thereby eliminating such a pattern. This idea follows directly from the efficient market hypothesis. Consequently, assuming market efficiency (even weak form), we should not observe serial correlation in financial market data.

Testing for Serial Correlation

There are a variety of tests for serial correlation, but the most common are the **Durbin–Watson (DW) test** and the **Breusch–Godfrey (BG) test**. The DW test is a measure of autocorrelation and compares the squared differences of successive residuals with the sum of the squared residuals. However, the DW test is limiting because it applies only to testing for first-order serial correlation. The BG test is more robust because it can detect autocorrelation up to a pre-designated order p, where the error in period t is correlated with the error in period $t - p$. The steps and logic of the procedure are outlined in Exhibit 8.

Exhibit 8: Breusch–Godfrey Test for Serial Correlation



The null hypothesis of the BG test is that there is no serial correlation in the model's residuals up to lag p. As noted in Exhibit 8, to keep things simple for illustrative purposes, the order p in Step 2 is 1, so there is one lagged residual, u_{t-1}, as an independent variable. The alternative hypothesis is that the correlation of residuals for at least one of the lags is different from zero and serial correlation exists up to that order. The test statistic is approximately *F*-distributed with $n - p - k - 1$ and p degrees of freedom, where p is the number of lags. This *F*-statistic is provided in most statistical software, so to evaluate the hypotheses, the resulting *P*-value can be compared with the desired level of significance.

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Suppose you want to assess the sensitivity of quarterly changes in GDP to changes in personal consumption expenditures (CONS). You use 30 years of quarterly data and estimate a model with the results shown in Exhibit 9, Panel A.

The coefficient of CONS is different from zero at the 1% significance level, as indicated by the *t*-statistic and the corresponding *P*-value. But because this is a time-series model, the analyst must also assess the presence of serial correlation in the regression residuals, especially given the significant *F*-statistic for the BG test in Panel B.

Exhibit 9: Regression of GDP on Consumer Expenditures

Panel A Regression Results

Regression Statistics	
Multiple R	0.946
R Squared	0.896
Adjusted R Squared	0.895
F-statistic	1023.0
Prob. of *F*-statistic	0.000
Observations	121

	Coefficients	***Standard Error***	***t Stat***	***P-value***	***Lower 95%***	***Upper 95%***
Intercept	0.0991	0.050	1.974	0.051	0.000	0.198
CONS	0.8696	0.027	31.990	0.000	0.816	0.923

Panel B Breusch–Godfrey Test for Serial Correlation

	Test Statistic	***P-value***
F-test	23.6113	0.0230

CODE: BREUSCH–GODFREY TEST

Using Python

```
import statsmodels.api as sm
from statsmodels.formula.api import ols
from statsmodels.graphics.tsaplots import plot_acf
df = pd.read_csv("data.csv")
model = ols('RETRF ~ MKTRF',data=df).fit()
print(model.summary())
r = model.resid
results = sm.stats.diagnostic.acorr_breusch_godfrey(model,nlags=10)
```

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```
print(results)
```

Using R

```
library(lmtest)
df <- read.csv("data.csv")
model <- lm(df$RETRF~df$MKTRF)
summary(model)
bgtest(model,order=10,type=c("Chisq","F"))
r <- model$res
```

Correcting for Serial Correlation

The most common "fix" for a regression with significant serial correlation is to adjust the coefficient standard errors to account for the serial correlation. Methods for adjusting standard errors are standard in many software packages. The corrections are known by various names, including **serial-correlation consistent standard errors**, *serial correlation and heteroskedasticity adjusted standard errors*, *Newey–West standard errors*, and *robust standard errors*. An advantage of these methods is that they also correct for conditional heteroskedasticity. The robust standard errors, for example, use heteroskedasticity- and autocorrelation-consistent (HAC) estimators of the variance–covariance matrix in the regression estimation.

Exhibit 10 shows the results of correcting standard errors from the regression of GDP on CONS. The coefficients for both the intercept and slope are unchanged. However, the robust standard errors are larger than the original OLS standard errors, so the *t*-statistics are now smaller and the *P*-values are larger. The key point is serial correlation in the regression error caused OLS to underestimate the uncertainty about the estimated parameters. Also, serial correlation is not eliminated, but the standard errors now account for it.

Exhibit 10: Regression of GDP on Consumption Expenditures with Robust Standard Errors (Correction for Serial Correlation)

Regression Statistics	
Multiple R	0.946
R Squared	0.896
Adjusted R Squared	0.895
F-statistic	591.0
Prob. of *F*-statistic	0.000
Observations	121

	Coefficients	***Standard Error***	***t Stat***	***P-value***	***Lower 95%***	***Upper 95%***
Intercept	0.0991	0.058	1.701	0.091	–0.016	0.214
CONS	0.8696	0.036	24.310	0.000	0.799	0.940

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As a reminder, correcting for serial correlation and heteroskedasticity is important for performing meaningful statistical tests. However, market efficiency implies these conditions should not arise in financial market data. If serial correlation and/or heteroskedasticity are observed, then discernible patterns in the fitted residuals contain information that has the potential to be exploited before they are eliminated by the trading activities of other market participants.

KNOWLEDGE CHECK

The senior analyst provides you, the junior analyst, with the following table for various multiple regression models he has estimated and then asks you to do the following:

1. Determine the critical *F*-value and correct conclusion for BG tests using a 5% significance level.

Model	Breusch–Godfrey *F*-statistic	Degrees of freedom	Critical *F*-value	Conclusion: Is there evidence of serial correlation of residuals?
A	5.1634	4, 13		
B	22.0560	6, 15		
C	2.3400	4, 15		
D	1.9800	3, 35		

Solution

Conclusions for each model based on the comparison of the BG statistic and the correct critical *F*-value are as follows:

	Breusch–Godfrey *F*-statistic	Degrees of freedom	Critical *F*-value	Conclusion: Is there evidence of serial correlation of residuals?
A	5.1634	4, 13	3.1791	Yes
B	22.0560	6, 15	2.7905	Yes
C	2.3400	4, 15	3.0556	No
D	1.9800	3, 35	2.8742	No

2. Describe robust standard errors and why they are useful.

Solution

Robust standard errors are regression coefficient standard errors that are corrected for possible bias arising from autocorrelation and heteroskedasticity. They are larger than OLS standard errors and allow the regression model results to be used for statistical inference.

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6 VIOLATIONS OF REGRESSION ASSUMPTIONS: MULTICOLLINEARITY

☐ explain multicollinearity and how it affects regression analysis

An assumption of multiple linear regression is that there is no exact linear relationship between two or more independent variables. When this assumption is violated, it becomes impossible to estimate the regression. However, **multicollinearity** may occur when two or more independent variables are highly correlated or when there is an approximate linear relationship among independent variables. With multicollinearity, the regression can be estimated, but interpretation of the role and significance of the independent variables is problematic. Multicollinearity is a serious concern because approximate linear relationships among economic and financial variables are common.

Consequences of Multicollinearity

Multicollinearity does not affect the consistency of regression coefficient estimates, but it makes these estimates imprecise and unreliable. Moreover, it becomes impossible to distinguish the individual impacts of the independent variables on the dependent variable. These consequences are reflected in inflated standard errors and diminished *t*-statistics, so *t*-tests of coefficients have little power (ability to reject the null hypothesis).

Detecting Multicollinearity

Except in the case of exactly two independent variables, using the magnitude of pairwise correlations among the independent variables to assess multicollinearity is generally inadequate. With more than two independent variables, high pairwise correlations are not a necessary condition for multicollinearity. For example, despite low pairwise correlations, there may be approximate linear combinations among several independent variables (which are unobservable) and that themselves are highly correlated.

The classic symptom of multicollinearity is a high R^2 and significant *F*-statistic but *t*-statistics for the individual estimated slope coefficients that are not significant due to inflated standard errors. While the coefficient estimates may be very imprecise, the independent variables as a group may do a good job of explaining the dependent variable.

Fortunately, we can use the **variance inflation factor (VIF)** to quantify multicollinearity issues. In a multiple regression, a VIF exists for each independent variable. Suppose we have k independent variables $X_1, \ldots, X_k$. By regressing one independent variable (X_j) on the remaining $k - 1$ independent variables, we obtain R_j^2 for the regression—the variation in X_j explained by the other $k - 1$ independent variables—from which the VIF for X_j is

$$VIF_j = \frac{1}{1 - R_j^2}. \tag{2}$$

For a given independent variable, X_j, the minimum VIF_j is 1, which occurs when R_j^2 is 0, so when there is no correlation between X_j and the remaining independent variables. VIF increases as the correlation increases; the higher the VIF, the more likely

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a given independent variable can be accurately predicted from the remaining independent variables, making it increasingly redundant. The following are useful rules of thumb:

- $VIF_j > 5$ warrants further investigation of the given independent variable.
- $VIF_j > 10$ indicates serious multicollinearity requiring correction.

IDENTIFYING MULTICOLLINEARITY AS A PROBLEM

Consider an analyst who is researching Fidelity Select Technology Portfolio (FSPTX), a mutual fund specializing in technology stocks. She wants to know if the fund behaves more like a large-cap growth fund or a large-cap value fund, so she estimates the following regression using 60 months of data:

$$FSPTX_t = b_0 + b_1 SGX_t + b_2 SVX_t + \varepsilon_t,$$

Where

$FSPTX_t$ is the monthly return to the Fidelity Select Technology Portfolio

SGX_t is the monthly return to the S&P 500 Growth Index

SVX_t is the monthly return to the S&P 500 Value Index

The regression results in Exhibit 11 indicate that the coefficients of SGX and SVX are different from zero at the 1% and 5% levels, respectively, implying the returns to the FSPTX fund are associated with returns to the growth index and returns to the value index.

Exhibit 11: Results of Regressing FSPTX Returns on Returns of the S&P 500 Growth and Value Indexes

Regression Statistics	
Multiple *R*	0.884
R-Squared	0.782
Adjusted *R*-Squared	0.774
Standard Error	0.027
Observations	60

ANOVA

	df	*SS*	*MS*	*F*	*Significance F*
Regression	2	0.1486	0.0743	102.2425	0.0000
Residual	57	0.0414	0.0007		
Total	59	0.1900			

	Coefficients	*Standard Error*	*t-Stat.*	*P-Value*	*Lower 95%*	*Upper 95%*
Intercept	–0.0069	0.004	–1.896	0.063	–0.014	0.000
SGX	1.7765	0.196	9.064	0.000	1.384	2.169
SVX	–0.4488	0.196	–2.292	0.026	–0.841	–0.057

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Suppose the analyst runs another regression, adding returns to the S&P 500 Index (SPX) to the model with SGX and SVX. Importantly, the S&P 500 Index includes the component stocks of these two style indexes (large-cap growth and value), so the analyst is inadvertently introducing severe multicollinearity and is over-specifying the model.

The results of the new regression are shown in Panel A of Exhibit 12. While the adjusted R^2 is little changed, now standard errors of the coefficients are higher: 6.166 for SGX and 5.503 for SVX versus 0.196 for both regressors in the prior model. Adding SPX returns does not explain any more of the variance in FSPTX fund returns, but now the coefficients for SGX and SVX are no longer statistically significant. This situation represents classic multicollinearity. We can visualize this in Panel B, with the correlation matrix representing the pairwise correlations between the variables.

Exhibit 12: Results of Regressing FSPTX Returns on Returns to the S&P 500 Growth and Value Indexes and the S&P 500 Index

Panel A Regression Results with SGX, SVX, and SPX

Regression Statistics	
Multiple *R*	0.884
R-Squared	0.782
Adjusted *R*-Squared	0.770
Standard Error	0.027
Observations	60

ANOVA

	df	***SS***	***MS***	***F***	***Significance F***
Regression	3	0.1486	0.0495	66.9683	0.0000
Residual	56	0.0414	0.0007		
Total	59	0.1900			

	Coefficients	***Standard Error***	***t-Stat.***	***P-Value***	***Lower 95%***	***Upper 95%***
Intercept	−0.0070	0.004	−1.877	0.066	−0.014	0.000
SGX	1.5302	6.166	0.248	0.805	−10.822	13.883
SVX	−0.6686	5.503	−0.121	0.904	−11.693	10.356
SPX	0.4658	11.657	0.040	0.968	−22.887	23.818

Panel B Correlation Matrix of Variables

	FSPTX	**SPX**	**SVX**	**SGX**
FSPTX	1	0.81	0.68	0.87
SPX	0.81	1	0.96	0.97

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	FSPTX	SPX	SVX	SGX
SVX	0.68	0.96	1	0.87
SGX	0.87	0.97	0.87	1

To understand the size of the multicollinearity problem, the analyst may compute VIFs for each independent variable in both regressions, as shown in Exhibit 13. Clearly, all three variables in the regression with SPX have large VIFs, indicating the error variances of their estimated coefficients are highly inflated and these variables are highly correlated. In contrast, VIFs for the coefficients of the two independent variables in the regression without SPX are both less than 5, suggesting multicollinearity is likely not a concern.

Exhibit 13: Variance Inflation Factors and Multicollinearity Problem

Model Explaining FSPTX Returns

With S&P 500		Without S&P 500	
Independent Variable	**VIF**	**Independent Variable**	**VIF**
SGX	3,863.85	SGX	3.97
SVX	3,083.22	SVX	3.97
SPX	12,904.81	–	

Correcting for Multicollinearity

Possible solutions to multicollinearity include

- excluding one or more of the regression variables,
- using a different proxy for one of the variables, and
- increasing the sample size.

Often, however, there is no easy solution for handling multicollinearity. So, you must experiment with including or excluding different independent variables to determine the source of and best solution to multicollinearity. However, if your goal is simply to use the model to predict the dependent variable—rather than to understand the roles of the independent variables—then multicollinearity may not be a major issue for you.

KNOWLEDGE CHECK

Returning to the example of the three-feature (CAPEX, ADV, R&D) ROA model for diversified manufacturers, for which the adjusted R^2 is 86.50% and the overall F-statistic is 54.4039 (partial ANOVA results shown below), the senior analyst is concerned about potential multicollinearity. So, she asks you, the junior analyst, to estimate VIF for each feature and then do the following:

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Partial ANOVA Results

	Coefficient	*Standard Error*	*t-Stat.*	*P-Value*
Intercept	4.7022	1.2349	3.8078	0.0010
CAPEX	1.2302	0.1056	11.6490	0.0000
ADV	–0.0371	0.1062	–0.3490	0.7304
R&D	0.1029	0.2837	0.3628	0.7203

Variable	**VIF**
CAPEX	1.164
ADV	1.068
RD	1.095

1. Determine whether multicollinearity is a concern using the regression results alone.

   Solution

   The overall *F*-statistic is significant, indicating at least one slope coefficient is non-zero. This is consistent with CAPEX's highly significant *t*-statistic, so multicollinearity should not be a concern.

2. Determine whether multicollinearity is a concern using the VIF results.

   Solution

   VIF for all features is well below 5 (closer to 1), indicating variances are not inflated, *t*-statistics are reliable, and multicollinearity is not an issue.

Exhibit 14 provides a summary of the violations of the assumptions of multiple linear regression that we have covered, the issues that result, and how to detect and mitigate them.

Exhibit 14: Summary of Violations of Assumptions from Model Misspecification

Assumption	Violation	Issue	Detection	Correction
Homoskedastic error terms	Heteroskedastic error terms	Biased estimates of coefficients' standard errors	Visual inspection of residuals; Breusch–Pagan test	Revise model; use robust standard errors
Independence of observations	Serial correlation	Inconsistent estimates of coefficients and biased standard errors	Breusch–Godfrey test	Revise model; use serial-correlation consistent standard errors
Independence of independent variables	Multicollinearity	Inflated standard errors	Variance inflation factor	Revise model; increase sample size

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PRACTICE PROBLEMS

The following information relates to questions 1-4

You are a junior analyst at an asset management firm. Your supervisor asks you to analyze the return drivers for one of the firm's portfolios. She asks you to construct three regression models of the portfolio's monthly excess returns (RET), starting with the following factors: the market excess return (MRKT), a value factor (HML), and the monthly percentage change in a volatility index (VIX). Next you add a size factor (SMB), and finally you add a momentum factor (MOM). Your three models are as follows:

Model 1: $\text{RET}_i = b_0 + b_{MRKT}\text{MRKT}_i + b_{HML}\text{HML}_i + b_{VIX}\text{VIX}_i + \varepsilon_i$.

Model 2: $\text{RET}_i = b_0 + b_{MRKT}\text{MRKT}_i + b_{HML}\text{HML}_i + b_{VIX}\text{VIX}_i + b_{SMB}\text{SMB}_i + \varepsilon_i$.

Model 3: $\text{RET}_i = b_0 + b_{MRKT}\text{MRKT}_i + b_{HML}\text{HML}_i + b_{VIX}\text{VIX}_i + b_{SMB}\text{SMB}_i + b_{MOM}\text{MOM}_i + \varepsilon_i$.

Your supervisor is concerned about conditional heteroskedasticity in Model 3 and asks you to perform the Breusch–Pagan (BP) test. At a 5% confidence level, the BP critical value is 11.07. You run the regression for the BP test; the results are shown in Exhibit 1.

Exhibit 1: Testing for Conditional Heteroskedasticity

Regression Statistics	
Multiple *R*	0.25517
R-Squared	0.06511
Adjusted *R*-Squared	0.01317
Standard Error	18.22568
Observations	96

Now the chief investment officer (CIO) joins the meeting and asks you to analyze two regression models (A and B) for the portfolio he manages. He gives you the test results for each of the models, shown in Exhibit 2.

Exhibit 2: Breusch–Godfrey and Durbin-Watson Test Results

	Test Type	Test Statistic	Critical Value	Independent Variable Is Lagged Value of Dependent Variable
Model A	Breusch–Godfrey	12.124	3.927	Yes
Model B	Durbin–Watson	5.088	4.387	No

The CIO also asks you to test a factor model for multicollinearity among its four explanatory variables. You calculate the variance inflation factor (VIF) for each of

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the four factors; the results are shown in Exhibit 3.

Exhibit 3: Multicollinearity Test Results

Variable	R^2	VIF
X_1	0.748	3.968
X_2	0.451	1.820
X_3	0.942	17.257
X_4	0.926	13.434

1. Calculate the BP test statistic using the data in Exhibit 1 and determine whether there is evidence of heteroskedasticity.

   A. 1.264, so there is no evidence of heteroskedasticity

   B. 6.251, so there is no evidence of heteroskedasticity

   C. 81.792, so there is evidence of heteroskedasticity

2. Identify the type of error and its impacts on regression Model A indicated by the data in Exhibit 2.

   A. Serial correlation, invalid coefficient estimates, and deflated standard errors.

   B. Heteroskedasticity, valid coefficient estimates, and deflated standard errors.

   C. Serial correlation, valid coefficient estimates, and inflated standard errors.

3. Determine using Exhibit 3 which one of the following statements is *most likely* to be correct. Multicollinearity issues exist for variables:

   A. X1 and X2.

   B. X2 and X3.

   C. X3 and X4.

4. Identify the correct answer related to the following statement.
   Possible solutions for addressing the multicollinearity issues identified in Exhibit 3 include:
   1. excluding one or more of the regression variables.
   2. using a different proxy for one of the variables.
   3. increasing the sample size.

   A. Only Solution 1 is correct.

   B. Only Solution 2 is correct.

   C. Solutions 1, 2, and 3 are each correct.

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SOLUTIONS

1. B is correct. The BP test statistic is calculated as nR^2, where n is the number of observations and R^2 is from the regression for the BP test. So, the BP test statistic = 96 × 0.06511 = 6.251. This is less than the critical value of 11.07, so we cannot reject the null hypothesis of no heteroskedasticity. Thus, there is no evidence of heteroskedasticity.

2. A is correct. The Breusch–Godfrey (BG) test is for serial correlation, and for Model A, the BG test statistic exceeds the critical value. In the presence of serial correlation, if the independent variable is a lagged value of the dependent variable, then regression coefficient estimates are invalid and coefficients' standard errors are deflated, so t-statistics are inflated.

3. C is correct. A VIF above 10 indicates serious multicollinearity issues requiring correction, while a VIF above 5 warrants further investigation of the given variable. Since X3 and X4 each have VIFs above 10, serious multicollinearity exists for these two variables. VIFs for X1 and X2 are both well below 5, so multicollinearity does not appear to be an issue with these variables.

4. C is correct. Possible solutions for addressing multicollinearity issues include all of the solutions mentioned: excluding one or more of the regression variables, using a different proxy for one of the variables, and increasing the sample size.

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LEARNING MODULE

4

Extensions of Multiple Regression

LEARNING OUTCOMES

Mastery	*The candidate should be able to:*
☐	describe influence analysis and methods of detecting influential data points
☐	formulate and interpret a multiple regression model that includes qualitative independent variables
☐	formulate and interpret a logistic regression model

INTRODUCTION 1

LEARNING MODULE OVERVIEW

- Two kinds of observations may potentially influence regression results: (1) a high-leverage point, an observation with an extreme value of an independent variable, and (2) an outlier, an observation with an extreme value of the dependent variable.
- A measure for identifying a high-leverage point is leverage, the distance of an extreme independent variable from the average of the independent variable. If leverage is greater than $3\left(\frac{k+1}{n}\right)$, where k is the number of independent variables and n is the number of observations, then the observation is potentially influential. A measure for identifying an outlier is studentized residuals, residuals from a model divided by its adjusted standard error. If the studentized residual is greater than a certain critical value of the t-statistic with $(n - k - 2)$ degrees of freedom, then the observation is potentially influential.
- Dummy, or indicator, variables represent qualitative independent variables and take a value of 1 (for true) or 0 (for false) to indicate whether a specific condition applies, such as whether a company belongs to a certain industry sector. To capture n possible categories, the model must include $(n - 1)$ dummy variables.
- An intercept dummy adds to or reduces the original intercept if a specific condition is met. When the intercept dummy is 1, the regression line shifts up or down parallel to the base regression line.

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- A slope dummy allows for a changing slope if a specific condition is met. When the slope dummy is 1, the slope changes to $(d_j + b_j) \times X_j$, where d_j is the coefficient on the dummy variable and b_j is the slope of X_j in the original regression line.
- A logistic regression model is one with a qualitative (i.e., categorical) dependent variable, so logistic regression is often used in binary classification problems, which are common in machine learning and neural networks.
- To estimate a logistic regression, the logistic transformation of the event probability (P) into the log odds, $\ln[P/(1 - P)]$, is applied, which linearizes the relation between the transformed dependent variable and the independent variables.
- Logistic regression coefficients are typically estimated using the maximum likelihood estimation (MLE) method, and slope coefficients are interpreted as the change in the log odds that the event happens per unit change in the independent variable, holding all other independent variables constant.

2 INFLUENCE ANALYSIS

☐ describe influence analysis and methods of detecting influential data points

Besides violations of regression assumptions, there is the issue that a small number of observations in a sample could potentially influence and bias regression results. An **influential observation** is an observation whose inclusion may significantly alter regression results. We discuss how to detect them and how to determine whether they do influence regression results.

Influential Data Points

Two kinds of observations may potentially influence regression results:

- A **high-leverage point**, a data point having an extreme value of an independent variable
- An **outlier**, a data point having an extreme value of the dependent variable

Both are substantially different from the majority of sample observations, but each presents itself in different ways.

Exhibit 1 shows a high-leverage point (triangle) in a sample of observations. Its X value does not follow the trend of the other observations; rather, it has an unusually high, possibly extreme, X value relative to the other observations. This observation should be investigated to determine whether it is an influential high-leverage point. Also, note the two estimated regression lines: The dashed line includes the high-leverage point in the regression sample; the solid line deletes it from the sample.

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Exhibit 1: Illustration of High-Leverage Point

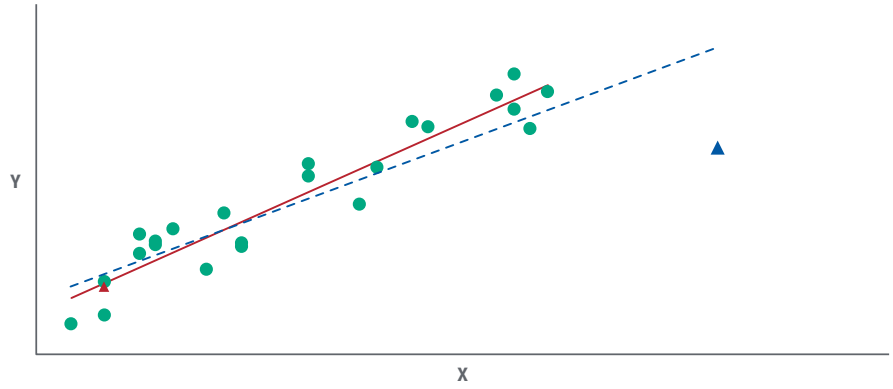
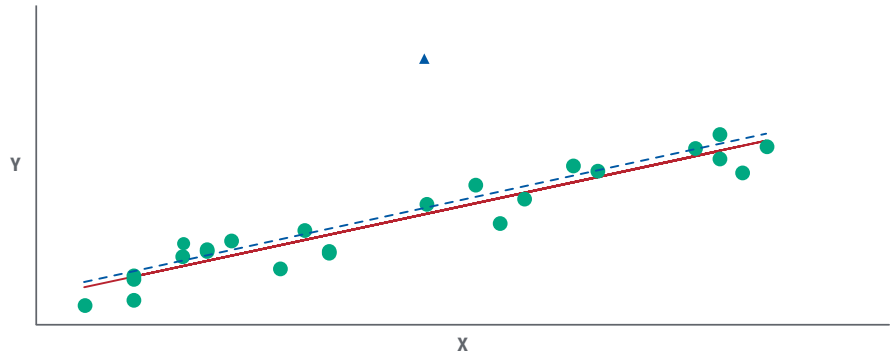


Exhibit 2 shows an outlier data point (triangle) in a sample of observations. Its Y value does not follow the trend of the other observations; rather, it has an unusual, possibly extreme, Y value relative to its predicted value, $\hat{Y}$, resulting in a large residual, $(Y - \hat{Y})$. This observation should be investigated to determine whether it is an influential outlier. Also, note the two estimated regression lines: The dashed line includes the outlier in the regression sample; the solid line deletes it from the sample.

Exhibit 2: Illustration of Potentially Influential Outlier



Outliers and high-leverage points are unusual but not necessarily a problem. For instance, a high-leverage point may deviate substantially from the other observations (in terms of values of the independent variable), but it may still lay close to the (solid) regression line. Problems arise if the high-leverage point or the outlier point are distant from the regression line. In these cases, the effect of the extreme value is to "tilt" the estimated regression line toward it, affecting slope coefficients and goodness-of-fit statistics.

Detecting Influential Points

A scatterplot is a straightforward way to identify outliers and high-leverage points in simple linear regression. However, multiple linear regression requires a quantitative way to measure the extreme values to reliably identify influential observations.

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A high-leverage point can be identified using a measure called **leverage** (h_{ii}). For a particular independent variable, leverage measures the distance between the value of the ith observation of that independent variable and the mean value of that variable across all n observations. Leverage is a value between 0 and 1, and the higher the leverage, the more distant the observation's value is from the variable's mean and, hence, the more influence the ith observation can potentially exert on the estimated regression.

The sum of the individual leverages for all observations equals $k + 1$, where k is the number of independent variables and 1 is added for the intercept. A useful rule of thumb for the leverage measure is that if an observation's leverage exceeds $3\left(\frac{k+1}{n}\right)$, then it is a potentially influential observation. Software packages can easily calculate the leverage measure.

KNOWLEDGE CHECK

Given the broad themes of "health consciousness" and "aging population," a senior specialty retail analyst tasks you, a junior analyst, with initiating coverage of nutritional supplement retailers. You begin by analyzing a cross-sectional dataset of 15 such specialty retailers to determine the impact of the number of their unique products (PROD)—such as vitamins, probiotics, antioxidants, and joint supplements—and the percentage of their online sales (ONLINE) on operating profit margins (OPM).

Your partial regression results are shown in Panel A of Exhibit 3, the leverage measure for each observation (i.e., company) is presented in Panel B, and your revised regression results after removing an observation from the sample are shown in Panel C.

Exhibit 3: Regression of Operating Profit Margin and Statistical Leverage Measure

Panel A OPM Regression Results with Full Sample

	Coefficient	Standard Error	*t*-Statistic	*P*-Value
Intercept	7.03	13.60	0.52	0.61
PROD	0.77	0.28	2.70	0.02
ONLINE	−0.29	0.28	−1.02	0.33

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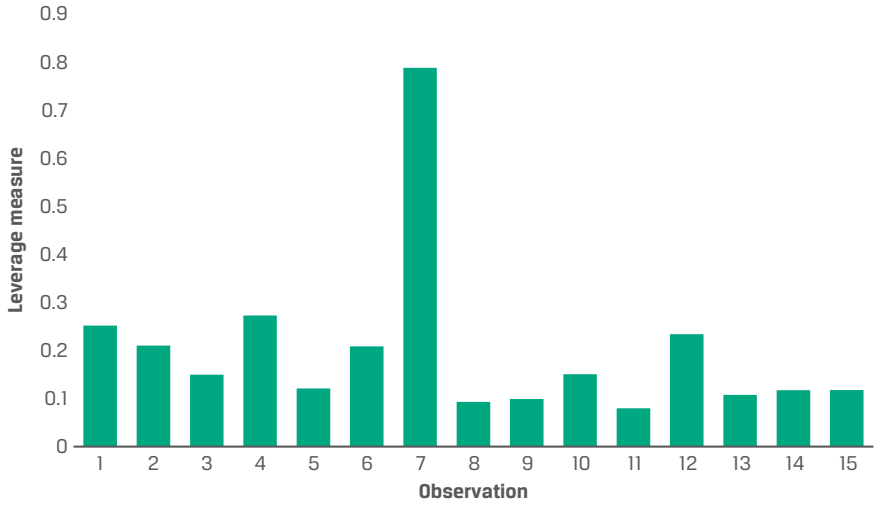
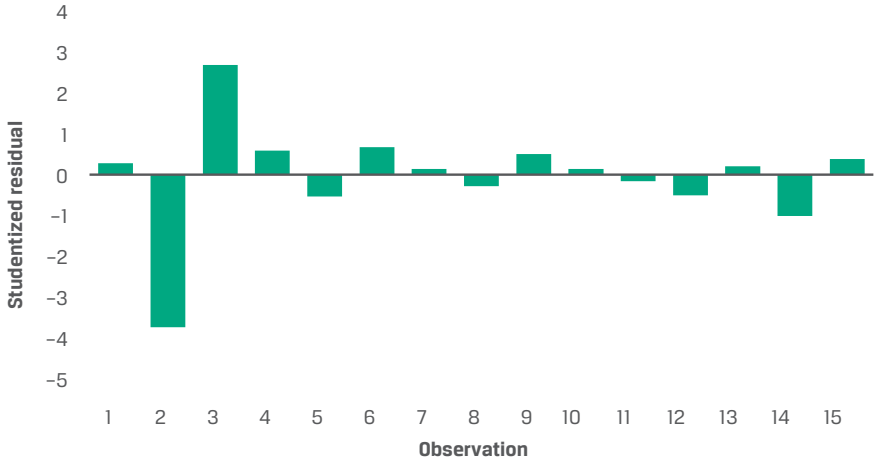
Panel B Leverage Measure





Panel C OPM Regression Results after Removing One Sample Observation

	Coefficient	Standard Error	*t*-Statistic	*P*-Value
Intercept	9.81	24.80	0.40	0.70
PROD	0.74	0.33	2.25	0.05
ONLINE	–0.36	0.61	–0.59	0.57

1. Identify which independent variables from the results in Panel A, if any, have slope coefficients different from zero at the 5% significance level.

   Solution:

   Only PROD has a slope coefficient different from zero at the 5% significance level; more specifically, it has a *P*-value of 0.02, or 2%.

2. Identify which observation(s) may be influential based on leverage, and justify your answer.

   Solution:

   Just one observation, Observation 7, is potentially influential. It has a leverage measure of 0.80, which exceeds the rule of thumb value,

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$3\left(\frac{k+1}{n}\right) = 3 \times \frac{2+1}{15} = \frac{9}{15} = 0.60$, for flagging possible influential data points.

3. Discuss how the regression results change after removing the potentially influential observation from the dataset.

Solution:

After removing Observation 7, the revised regression results show a larger intercept (9.81) and reduced significance for PROD, now with a *P*-value of 5%. However, while this observation may be considered influential, the full sample conclusion—PROD's coefficient is different from zero at the 5% significance level—still holds.

As for identifying outliers, observations with unusual dependent variable values, the preferred method is to use **studentized residuals**. The logic and process behind this method are as follows:

1. Estimate the initial regression model with n observations, then sequentially delete observations, one at a time, and each time re-estimate the regression model on the remaining $(n - 1)$ observations.
2. Compare the observed Y values (on n observations) with the predicted Y values resulting from the models with the ith observation deleted—on $(n - 1)$ observations. For a given observation i, the difference (or residual) between the observed Y_i and the predicted Y with the ith observation deleted $\left(e_i^*\right)$ is $e_i^* = Y_i - \hat{Y}_{i^*}$.
3. Divide this residual by the estimated standard deviation or standard error of the residuals, s_e^*, which produces the **studentized deleted residual**, t_{i^*}:

$$t_{i^*} = \frac{e_i^*}{s_{e^*}} = \frac{e_i}{\sqrt{\text{MSE}_{(i)}(1 - h_{ii})}} \quad (1)$$

Also, note the equivalent formula (on the right) for t_{i^*}, whose terms are all based on the initial estimated regression with n observations, where

e_i^* = the residual with the ith observation deleted

s_{e^*} = the standard deviation of all the residuals

k = the number of independent variables

$\text{MSE}_{(i)}$ = the mean squared error of the regression model that deletes the ith observation

h_{ii} = the leverage value for the ith observation

Studentized deleted residuals are effective for detecting influential outlying Y observations. Exhibit 4 presents rules of thumb for using them to flag outliers and the test to determine whether the outlier is influential. Automatically excluding one or several influential datapoints may have a significant impact on the statistical estimates, including the coefficient estimates and their possible interpretation.

Note the studentized residual value must be compared to the critical value of the t-distributed statistic with $(n - k - 2)$ degrees of freedom at the selected significance level to conclude whether the observation is potentially influential.

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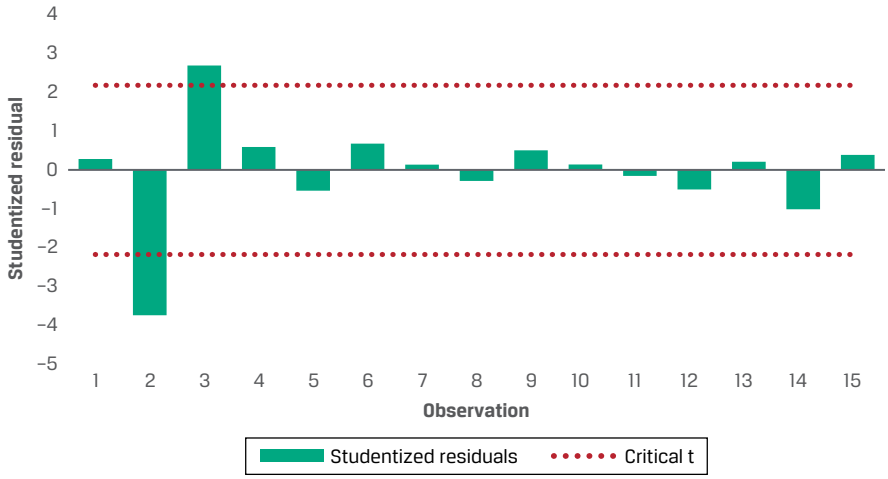
Exhibit 4: Using Studentized Residuals to Identify Influential Outliers

If …	Then …
$\lvert t_i^* \rvert > 3$	Flag observation as being an outlier
$\lvert t_i^* \rvert >$ critical value of t-statistic with $n - k - 2$ degrees of freedom at selected significance level	Flag outlier observation as being potentially influential

Consider the junior analyst testing the OPM regression model with two regressors (PROD and ONLINE) based on the sample of 15 nutritional supplement retailers. The analyst wants to detect outliers and, if any, determine whether they are influential. A visual of the studentized residuals for the regression model is provided in Exhibit 5. Note that the absolute value of the studentized residuals for Observations 2 and 3 exceeds the critical t-value of 2.2010 for 11, $(n - k - 2) = 15 - 2 - 2$, degrees of freedom, indicating they may be influential. Consequently, these observations should be flagged for further examination.

Exhibit 5: Detecting Influential Outliers Using Studentized Residuals



Outliers and high-leverage points are not necessarily influential. An observation is considered influential if its exclusion from the sample causes substantial changes in the estimated regression function.

The roles of leverage and studentized residuals in detecting influential data points are summarized in Exhibit 6. Although the calculation of these measures seems daunting, most statistical software packages readily provide them. The leverage and studentized residuals measures can reveal why a given observation is influential—that is, due to it having an extreme X value or an extreme Y value, respectively. Additionally, data visualization can reveal clues of influence.

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Exhibit 6: Summary of Measures of Influential Observations

	Influence of …			
Measure	**Dependent variable**	**Independent variable**	**Process**	**Is observation influential?**
Leverage		x	h_{ii} ranges from 0 to 1	If $h_{ii} > 3\left(\frac{k+1}{n}\right)$, then potentially influential
Studentized residual	x		Compare calculated $\lvert t\text{-statistic}\rvert$ with critical t-value	If calculated $\lvert t\text{-statistic}\rvert >$ critical t-value, then potentially influential

To summarize, leverage is key for detecting high-leverage data points and studentized residuals are key for revealing potentially influential outliers. Leverage and studentized residuals, with the influence plot for visualizing them, and the revised regression results should all be evaluated when the analyst's objective is detection of influential data points.

Besides detecting influential data points, we must investigate why they occur and determine a remedy. In some cases, an influential data point is simply due to data input errors or inaccurate measurements. The remedy is to either correct the erroneous data or discard them and then to re-estimate the regression using the cleansed sample. Alternatively, the dataset can also be winsorized to mitigate the impact of outliers found in the dataset. In other cases, the influential data points are valid, which may indicate that important explanatory variables are omitted from the model or regression assumptions are being violated. We must resolve these issues by identifying and including potentially useful explanatory variables and/or checking that our model satisfies all regression assumptions; effectively, the modeling objective determines whether influential observations are outliers or integral to the data.

QUESTION SET

You are analyzing a regression model of companies' return on assets (ROA) estimated with 26 observations and three independent variables and are concerned about outliers and influential observations. Using software, you calculate the studentized residual t-statistic for each observation, as shown below.

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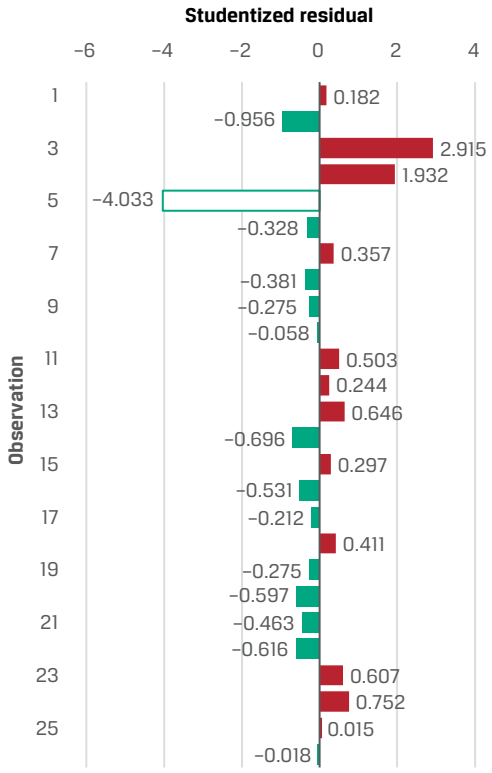

While meeting with the research director to discuss your results, she asks you to do the following:

1. Identify which observations, if any, are considered outliers using studentized residuals at a 5% significance level.

   Solution:

   There are 21 (= 26 – 3 – 2) degrees of freedom, so the critical *t*-statistics are ±2.080. Therefore, Observations 3 and 5 are considered outliers based on the studentized residuals.

2. Recommend what actions, if any, you should take to deal with any influential observations.

   Solution:

   From the results of studentized residuals , the analyst should investigate outlier Observations 3 and 5 to ensure there are no data entry or quality issues.

DUMMY VARIABLES IN A MULTIPLE LINEAR REGRESSION

3

☐ formulate and interpret a multiple regression model that includes qualitative independent variables

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Analysts often must use qualitative variables as independent variables in a regression. One such type of variable is a **dummy variable** (or *indicator variable*). A dummy variable takes on a value of 1 if a particular condition is true and 0 if that condition is false. A key purpose of using dummy variables is to distinguish between "groups" or "categories" of data.

Defining a Dummy Variable

A dummy variable may arise in several ways, including the following:

- It may reflect an inherent property of the data (i.e., industry membership).
- It may be a characteristic of the data represented by a condition that is either true or false (i.e., a date before or after a key market event).
- It may be constructed from some characteristic of the data where the dummy variable reflects a condition that is either true or false (i.e., firm sales less than or greater than some value).

We must be careful when choosing the number of dummy variables in a regression to represent a specific condition. If we want to distinguish among n categories, we need $n - 1$ dummy variables. So, if we use dummy variables to denote companies belonging to one of five industry sectors, we use four dummies, as shown in in Exhibit 7. The analysis still applies to five categories, but the category not assigned becomes the "base" or "control" group and the slope of each dummy variable is interpreted relative to the base. In this case, the base group is Food & Beverage.

Exhibit 7: Using Dummies to Represent Membership in Industry Sector

	Dummy Variables			
Industry Sector	**Technology**	**Financial Services**	**Health Care**	**Energy**
Technology	1	0	0	0
Financial Services	0	1	0	0
Health Care	0	0	1	0
Energy	0	0	0	1
Food & Beverage*	0	0	0	0

**Food & Beverage is the base (i.e., control) group.*

The reason for using $n - 1$ dummy variables is to avoid violating the assumption that no exact linear relationship exists between two or more independent variables. If we included dummy variables for all n categories, rather than $n - 1$, the regression would fail because the dummies would sum to the variable used to estimate the intercept in the regression.

Visualizing and Interpreting Dummy Variables

A common type of dummy variable is the **intercept dummy**. Consider a regression model for the dependent variable Y that involves one continuous independent variable, X, and one intercept dummy variable, D.

$$Y_i = b_0 + d_0 D + b_1 X_i + \varepsilon_i. \quad (2)$$

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This single regression model estimates two lines of best fit corresponding to the value of the dummy variable:

- If $D = 0$, then the equation becomes $Y = b_0 + b_1X + \varepsilon$ (*base category*).
- If $D = 1$, then the equation becomes $Y = (b_0 + d_0) + b_1X + \varepsilon$ (*category to which the changed intercept applies*).

Panel A of Exhibit 8 illustrates the effect of the intercept shift from a dummy variable; it is the vertical distance d_0. The shift can be positive or negative (here it is positive). The solid line where the dummy takes the value of zero ($D = 0$) relates to the base category; the parallel dashed line where the dummy variable takes the value of 1 ($D = 1$) relates to the category to which the dummy variable applies.

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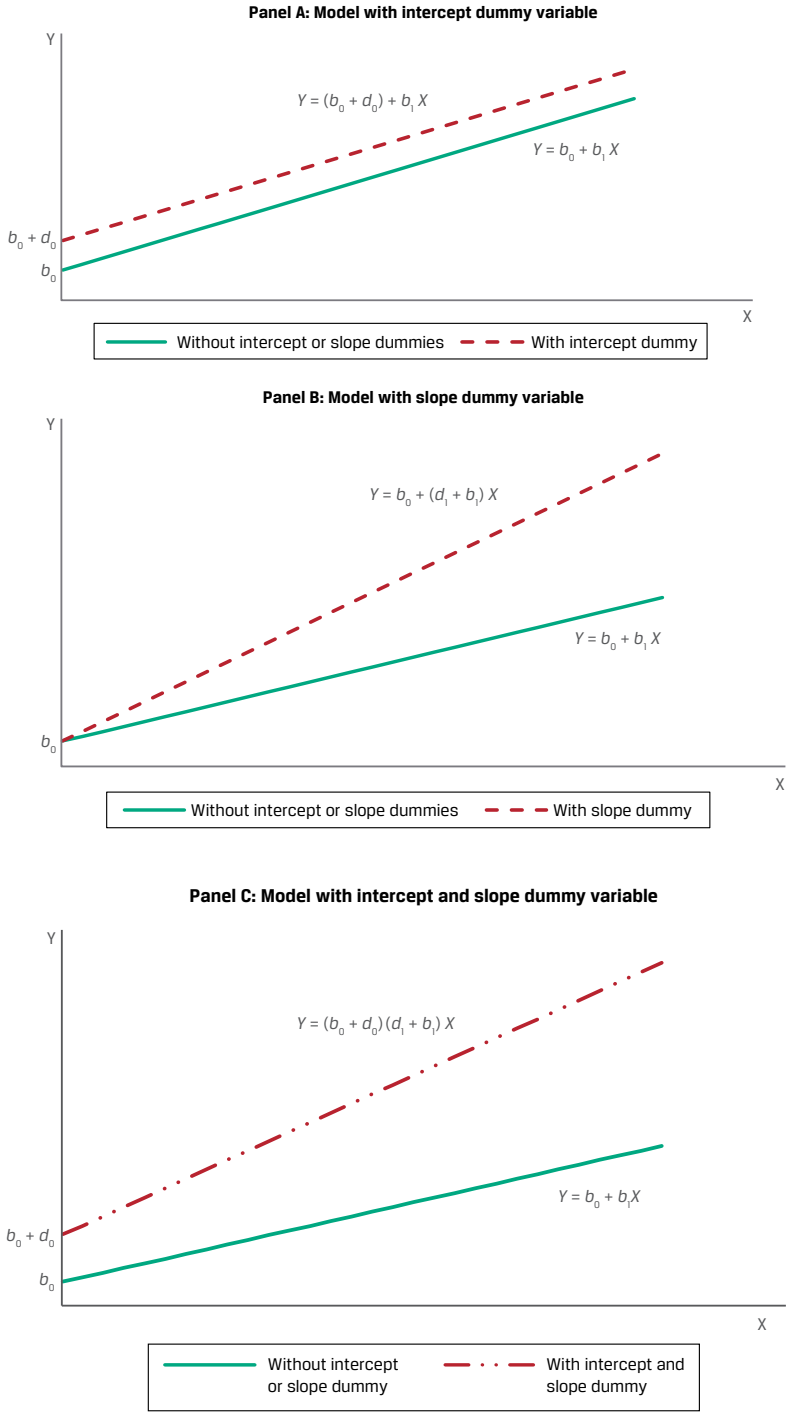
Exhibit 8: Visualizing Intercept and Slope Dummies



A different scenario uses a dummy that allows for slope differences, a **slope dummy**, which can be explained using a simple model with one continuous variable (X) and one slope dummy variable (D).

$$Y_i = b_0 + b_1 X_i + d_1 D_i X_i + \varepsilon_i. \quad (3)$$

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The slope dummy variable creates an **interaction term** between the X variable and the condition represented by $D = 1$. The slope dummy is interpreted as a change in the slope between the categories captured by the dummy variable:

- If $D = 0$, then $Y = b_0 + b_1X + \varepsilon$ (*base category*).
- If $D = 1$, then $Y = b_0 + (b_1 + d_1)\,X + \varepsilon$ (category to which changed slope applies).

As before, the case of $D = 0$ is the base category. The dummy variable allows for slopes to differ between the two categories. In Panel B of Exhibit 8, the base category is the same as before (shown by the solid line). For the other category, the relationship between Y and X is shown by the steeper-sloping dashed line for $Y = b_0 + (b_1 + d_1)X$. The difference between slopes may be positive or negative, depending on the scenario (here it is positive).

It is also possible for a regression to use dummies in both the slope and the intercept. To do so, we combine the two previous models.

$$Y_i - b_0 + d_0D_1 + b_1X_i + d_1D_iX_i + \varepsilon_i. \tag{4}$$

- If $D = 0$, then $Y = b_0 + b_1X + \varepsilon$ (*base category*).
- If $D = 1$, then $Y = (b_0 + d_0) + (b_1 + d_1)X + \varepsilon_i$ (*category to which both changed intercept and changed slope apply*).

This model allows for a change in both intercept and slope across the two groups, shown in Panel C of Exhibit 8 by the dashed line above the solid line. In this more complex treatment, the difference between the two categories depends on both an intercept effect (d_0) and a slope effect (d_1X) that varies with the size of the independent variable. Note in this scenario $d_1 > 0$. Finally, these scenarios are based on only two categories. We may have more categories with more dummies and more independent variables; in this case, the graphs would show more fitted lines, one relating to each category.

Testing for Statistical Significance of Dummy Variables

As explained, dummy variables are useful for distinguishing between categories of data. Tests of whether a regression function is different for one group versus another are straightforward with dummy variables. Individual t-tests on the dummy variable coefficients indicate whether they are significantly different from zero.

Exhibit 9 illustrates dummy variables in a regression using a cross-section of mutual fund data. An analyst has been tasked with analyzing how mutual fund characteristics affect fund returns. She uses a large database of mutual funds that includes several styles: blend, growth, and value.

The dependent variable is fund five-year average annual return. The independent variables are

- fund expense ratio (EXP),
- fund portfolio cash ratio (CASH),
- fund age (AGE), and
- the natural logarithm of fund size (SIZE).

Given three possible style categories, she uses n − 1 = 2 dummy variables:

- BLEND, which takes a value of 1 if the fund is a blend fund and 0 otherwise;
- GROWTH, which takes a value of 1 if the fund is a growth fund and 0 otherwise; and
- VALUE, the base category without a dummy.

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The classification for the dummy variables is shown in Panel A of Exhibit 9. The regression model is

$$\text{Returns}_i = b_0 + b_1\text{EXP}_i + b_2\text{CASH}_i + b_3\text{AGE}_i + b_4\text{SIZE}_i + d_1\text{BLEND}_i + d_2\text{GROWTH}_i + \varepsilon_i.$$

The regression output in Panel B of Exhibit 9 shows that all slope coefficients and the intercept are significantly different from zero. The R^2 estimate suggests that the empirical specification explains 12.30% of the variation in the observed five-year average annual return; its marginal difference from the adjusted R^2 estimate, 12.28%, indicates that the model specification builds on statistically significant variables. The dummy coefficients—0.66 for BLEND and 2.50 for GROWTH—suggest blend funds deliver average annual returns exceeding those of the value category by 0.66% while growth funds deliver 2.50% more than the base value category. Moreover, the intercept coefficient suggests that an average annual return of –2.91% is unexplained by the model's independent variables.

Exhibit 9: Analysis of Mutual Funds in Different Categories

Panel A Classification of Mutual Funds

Mutual Fund Style	Dummy Variable	
	Blend	Growth
Blend	1	0
Growth	0	1
Value*	0	0

**Value is the base (i.e., control) group.*

Panel B Explaining Mutual Fund Returns with Fund Type Dummies

Regression Statistics	
R^2	0.1230
Adjusted R^2	0.1228
Standard Error	4.224
Observations	23,025

ANOVA	Df	SS	MS	F	Significance F
Regression	6	57,636.46	9,606	538	0
Residual	23,018	410,816.9	17.85		
Total	23,024	468,453.3			

	Coefficient	Standard Error	*t*-Statistic	*P*-Value
Intercept	–2.909	0.2990	–9.738	0.00
EXP	–0.586	0.0495	–11.824	0.00
CASH	–0.032	0.0029	–11.168	0.00
AGE	0.074	0.0033	22.605	0.00
SIZE	0.267	0.0141	18.924	0.00
BLEND	0.661	0.0678	9.749	0.00
GROWTH	2.498	0.0748	33.394	0.00

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The analyst extends the study by adding slope dummies. Initial results suggests a small impact of fund age on returns, 0.07% per year of age. She wonders whether this relationship between age and performance differs by fund type. For example, does the age factor affect growth or blend funds differently from how it affects value funds? To explore this idea, she introduces two interaction variables—slope dummies, AGE_BLEND and AGE_GROWTH—and estimates the following model:

$$\text{Returns}_i = b_0 + b_1\text{EXP}_i + b_2\text{CASH}_i + b_3\text{AGE}_i + b_4\text{SIZE}_i + d_1\text{BLEND}_i + d_2\text{GROWTH}_i + d_3\text{AGE\_BLEND}_i + d_4\text{AGE\_GROWTH}_i + \varepsilon_i.$$

When BLEND = 1, the interaction term AGE_BLEND takes the value of AGE; otherwise, it takes the value of zero. Similarly, when GROWTH = 1, the interaction term AGE_GROWTH takes the value of AGE; otherwise, it takes the value of zero. Exhibit 10 presents the revised regression results.

Exhibit 10: Explaining Mutual Fund Returns with Intercept and Slope Dummies

Regression Statistics	
R^2	0.123
Adjusted R^2	0.123
Standard Error	4.224
Observations	23,025

ANOVA	Df	SS	MS	F	Significance F
Regression	8	57,760.46	7,220	404.6	0.000
Residual	23,016	410,692.9	17.84		
Total	23,024	468,453.3			

	Coefficient	Standard Error	*t*-Statistic	*P*-Value
Intercept	−2.810	0.3060	−9.183	0.00
EXP	−0.587	0.0496	−11.839	0.00
CASH	−0.032	0.0029	−11.211	0.00
AGE	0.065	0.0059	11.012	0.00
SIZE	0.267	0.0141	18.906	0.00
BLEND	0.603	0.1088	5.546	0.00
GROWTH	2.262	0.1204	18.779	0.00
AGE_BLEND	0.005	0.0077	0.627	0.53
AGE_GROWTH	0.020	0.0081	2.478	0.01

These results show that the values and significance of the slope coefficients are little changed. But the revised model provides more information on AGE. For the base group (value funds), the AGE coefficient suggests those funds earn an extra return of 0.065% as time passes. This is when BLEND = 0 and GROWTH = 0.

The interaction term AGE_GROWTH is statistically significant, with a *p*-value = 0.01, implying for growth funds an extra annual return with each year of age equal to the sum of the AGE and AGE_GROWTH coefficients, or 0.085% (= 0.065% + 0.020%). So, the "slope" coefficient for GROWTH (with respect to AGE) is the sum of those

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two coefficients. Finally, we can interpret the overall result as suggesting that growth funds' returns exceed those of value funds by 2.347%, or 2.262% (GROWTH) plus 0.085% (AGE + AGE_GROWTH), for each year of a fund's age (since inception).

QUESTION SET

You are interviewing for the position of junior analyst at a global macro hedge fund. The managing director (MD) interviewing you outlines the following scenario: You are tasked with studying the relation between stock market returns and GDP growth for multiple countries and must use a binary variable in your regression model to categorize countries by stock market type, emerging (1) or developed (0) markets. The MD provides the following three modeling choices:

A. Slope dummy: It allows for a change in slope to classify countries into weak stock performance countries and strong stock performance countries.

B. Intercept dummy: It allows for a change in intercept to classify countries by their stock market development status.

C. Interaction term: It allows for a change in intercept to classify countries into low-GDP growth and high-GDP growth countries.

1. To answer the MD's question, identify the appropriate new variable and its function.

   A. Slope dummy.

   B. Intercept dummy.

   C. Interaction term.

   Solution:

   B is correct. The new variable, an intercept dummy, allows for a change in intercept to classify countries by emerging versus developed stock market status.

The MD continues, indicating that you must refine the model to capture the effect on stock returns of the interaction of each country's GDP growth and its stock market development status. He then asks you to do the following:

2. Identify the model you should use (noting the following definitions).

   GDPG: Country GDP growth

   EM: Indicates emerging stock market country

   DM: Indicates developed stock market country

   A. Stock return = $b_0 + b_1\text{GDPG} + d_1\text{EM} + d_2\text{DM} + d_3(\text{EM} \times \text{GDPG}) + \varepsilon$.

   B. Stock return = $b_0 + b_1\text{GDPG} + d_1\text{EM} + d_2\text{DM} + \varepsilon$.

   C. Stock return = $b_0 + b_1\text{GDPG} + d_1\text{EM} + d_2(\text{EM} \times \text{GDPG}) + \varepsilon$.

   Solution:

   C is correct. This model includes a variable for country GDP growth (GDPG); one dummy for emerging stock market status (EM = 1, 0 otherwise), with developed market status as the base case; and a term (EM × GDPG) for the interaction of EM status with GDP growth.

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Another MD joins the interview and mentions that an analyst on her team estimated a regression to explain a cross-section of returns on assets of companies using a regulation dummy variable (REG = 1 if regulated, 0 otherwise), market share (MKTSH), and an interaction term, REG_MKTSH, the product of REG and MKTSH. She notes the resulting model is

$$RET = 0.50 - 0.5REG + 0.4MKTSH - 0.2REG\_MKTSH$$

and asks you to do the following:

3. Identify which of the following statements is *correct* regarding interpretation of the regression results (indicate all that apply).

   A. The average return for a regulated firm is at least 0.5% lower than for a non-regulated firm, holding the market share constant.

   B. Non-regulated companies with larger market shares have lower ROAs than regulated companies.

   C. For each increase in market share, a regulated firm will have an increasingly lower ROA than an unregulated firm.

   Solution:

   Both A and C are correct.

   A is correct because the coefficient on REG is –0.5. As MKTSH approaches 0, we see that the regulated firm has 0.5% less return. Or, if the Market Share Contribution to return is the same, that is, 0.2*MKTSH(Regulated) = 0.4*MKTSH(Non-regulated), then the regulated firm has 0.5% less return.

   C is correct because the sum of coefficients is –0.3 = –0.5REG + 0.4MKTSH –0.2REG_MKTSH). If MKTSH increases by 1%, for both regulated and non-regulated, the regulated firm will have a return that is 0.2% less, 0.2(1%) - 0.4(1%) = -0.2%. The 0.5% return of the non-regulated does not get included, since we are looking at the change in the return, based on a 1% increase in MKTSH.

   B is incorrect because the coefficient on MKTSH is positive and the coefficient on REG is negative.

MULTIPLE LINEAR REGRESSION WITH QUALITATIVE DEPENDENT VARIABLES

4

☐ formulate and interpret a logistic regression model

A **qualitative dependent variable** (categorical dependent variable) is an outcome variable describing data that fit into categories. For example, to predict whether a company will go bankrupt or not, we need a qualitative dependent variable (bankrupt or not bankrupt) and company financial performance data (e.g., return on equity, debt-to-equity ratio, or debt rating) as independent variables. In this example, the bankrupt or not bankrupt qualitative dependent variable is binary, but a dependent variable that falls into more than two categories is also possible.

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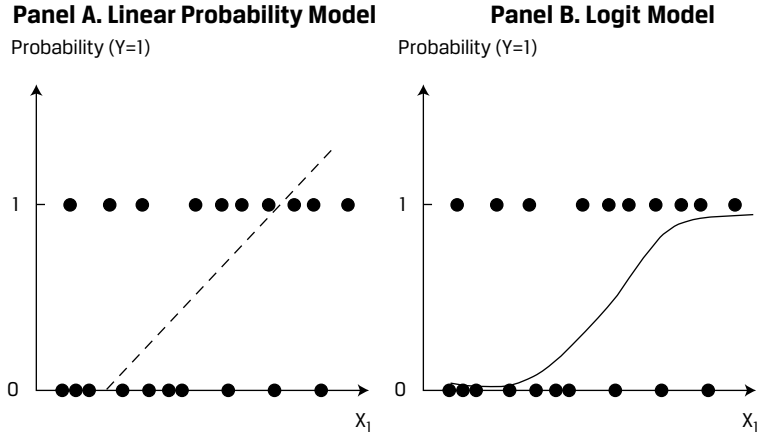
In contrast to a linear regression, the dependent variable here is not continuous but discrete (binary). Estimating such a model using linear regression is not appropriate. If we were to try to estimate this using the qualitative dependent variable, such as Y = 1 if bankrupt and 0 if not, in a linear model with three independent variables, then we would be estimating a linear probability model:

$$Y_i = b_0 + b_1X_{1i} + b_2X_{2i} + b_3X_{3i} + \varepsilon_i. \quad (5)$$

The problem with this form is that the predicted value of the dependent variable could be greater than 1 or less than 0, depending on the estimated coefficients b_i and the value of observed independent variables. Generating predicted values above 1.0 or below 0 would be invalid, because the probability of bankruptcy (or of anything) cannot be greater than 1.0 or less than 0. Moreover, linear regression assumes the relationship between the probability of bankruptcy and each financial variable is linear over the range of the financial variable, which might be unrealistic. For example, one can reasonably expect that the probability of bankruptcy and the debt-to-equity ratio are not linearly related for very low or high levels of that variable.

To address these issues, we apply a non-linear transformation to the probability of bankruptcy and relate the transformed probabilities linearly to the independent variables. The most commonly used transformation is the **logistic transformation**. Exhibit 11 shows the linear probability model (Panel A) and logistic regression model (Panel B), where the logit model's non-linear function takes on a sigmoidal shape and is approximately linear except when probability estimates are close to zero or one.

Exhibit 11: Linear Probability Model versus Logistic Regression Model



Let P be the probability of bankruptcy or, generally, that a condition is fulfilled or an event happens. The logistic transformation is

$$\ln\left(\frac{P}{1-P}\right). \quad (6)$$

The ratio $\left(\frac{P}{1-P}\right)$ is a ratio of probabilities—the probability that the event of interest happens (P) divided by the probability that it does not happen ($1 - P$), with the ratio representing the odds of an event happening.

For example, if the probability of a company going bankrupt is 0.75 and $P/(1 - P)$ is $0.75/(1 - 0.75) = 3$, the odds of bankruptcy are 3 to 1. This implies that the probability of bankruptcy is three times as large as the probability of the company not going bankrupt. The natural logarithm (ln) of the odds of an event happening is the **log odds**, which is also known as the *logit function*.

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Logistic regression (logit) uses the logistic transformation of the event probability (P) into the log odds, $\ln\left(\frac{P}{1-P}\right)$, as the dependent variable:

$$\ln\left(\frac{P}{1-P}\right) = b_0 + b_1X_1 + b_2X_2 + b_3X_3 + \varepsilon. \quad (7)$$

Once the log odds are estimated, the event probability can be derived as

$$P = \frac{1}{1+\exp\left[-(b_0 + b_1X_1 + b_2X_2 + b_3X_3)\right]}. \quad (8)$$

Logistic regression assumes a logistic distribution for the error term; the distribution's shape is similar to the normal distribution but with fatter tails.

Logistic regression coefficients are typically estimated using the **maximum likelihood estimation (MLE)** method rather than by least squares. The MLE method estimates logistic regression coefficients that make it most likely that the choices in the sample would occur by maximizing the likelihood function for the data. For logistic regression, the probability distribution used to construct the likelihood function for the data is the binomial distribution because each outcome is binary.

The prediction is binary (0 or 1), and the logistic regression function produces the probability (P) that the outcome is 1. MLE estimates of the intercept and slope coefficients in the logistic regression are the values that make the data in the regression sample "most likely" and are derived via statistical software. Notably, a hypothesis test that a logit regression coefficient is significantly different from zero is similar to the test in OLS regression.

For logistic regression, most statistical software packages produce the log-likelihood, a chi-square-distributed test statistic. The performance of a logit model in classifying the dependent variable (as 0 or 1) may be evaluated by examining the p-value of this test statistic versus selected significance levels.

Logistic regression does not have an equivalent measure to R^2 because it cannot be fitted using least squares. Pseudo-R^2 has been proposed to capture the explained variation in a logistic regression and is generated in standard software output. The pseudo-R^2 must be interpreted with care because it can only be used to compare different specifications of the same model (not models based on different datasets).

Determining the marginal effect of a change in a variable in the logistic model is not as straightforward as in a regression model. Unlike the linear OLS regression, a logistic regression is non-linear and the interpretation of the estimates for a logistic regression depends not only on the estimate for the specific variable estimate but also on the level of the other variable estimates. The non-linear nature of the relationship in the logistic regression estimate depicts the marginal contribution of each variable estimate to the slope of the probability curve. That is why the impact of a one-unit incremental change of an independent variable on the probability of $Y = 1$ depends on the level of all the other independent variables.

In a logistic regression, the change in the probability for a given change in a variable is the slope of the probability curve. The slope can be expressed as the derivative of probability. That probability contains both an exponential function and the derivative of that exponential function. The derivative is the exponential function itself multiplied by the derivative of the contents of the exponential function. Effectively in a logistic model, the value of the derivative changes depending on the slope and its relative position. In the linear probability model, however, the derivative is a constant; thus, the slope is constant and the marginal effect is a constant.

In Exhibit 12, it is clear that the impact of a one-unit change in X_1 on $P(Y = 1)$ will depend on the overall value of $\left(\hat{b}_0 + \hat{b}_1X_1 + \hat{b}_2X_2 + \hat{b}_3X_3\right)$. When $\left(\hat{b}_0 + \hat{b}_1X_1 + \hat{b}_2X_2 + \hat{b}_3X_3\right)$ is very small or very large for an observation, the impact of a one-unit change in X_1 on $P(Y = 1)$ will be small because the slope of $P(Y = 1)$ is

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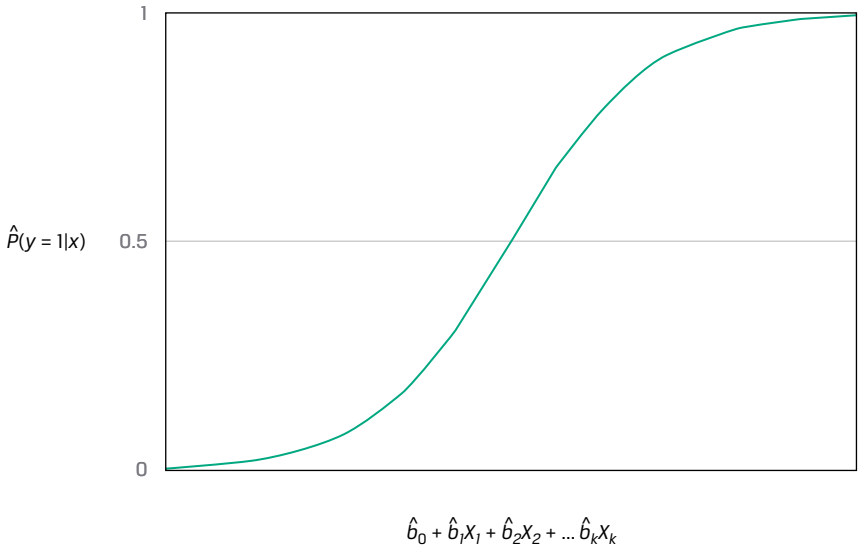
close to zero. However, when $\left(\hat{b}_0 + \hat{b}_1 X_1 + \hat{b}_2 X_2 + \hat{b}_3 X_3\right)$ is near the inflection point, $P(Y = 1) = 0.5$, a one-unit change in X_1 will have a larger impact on $P(Y = 1)$, since the slope of $P(Y = 1)$ is large.

Exhibit 12: Fitted Logistic Regression



One way to estimate this marginal effect of an independent variable is to find the marginal effect for the "average" company/investment/observation. This process uses the mean value in the dataset for each of the X_i's, calculates the probability using these mean values, and then calculates the probability again using the mean values for all variables except the independent variable of interest, which is increased from its mean value by one unit. This provides an estimate of the marginal effect of a change in the chosen independent variable on $P(Y = 1)$. Most software programs have an option to conduct this analysis. For instance in Python, "`sklearn, get_margeff`" can be used, and in R, in the margins package, "`margins`," can be used.

It is equally reasonable to examine the marginal effect for a particular observation—may it be associated with a company, an investment, or any other relevant observation—by first specifying the observation values and using these in the calculations outlined previously to calculate the marginal effect of the chosen variable based on its characteristics. Then this result provides an estimate of the marginal probability: how much the probability of the event changes should one or more of its characteristics change. Effectively, this approach provides a sensitivity analysis to incremental changes.

The **likelihood ratio (LR) test** is a method to assess the fit of logistic regression models and is based on the log-likelihood metric that describes the fit to the data. The LR test statistic is

$$\text{LR} = -2 \times (\text{Log-likelihood restricted model} - \text{Log-likelihood unrestricted model}).$$

The test is similar to the joint F-test of hypotheses used in least squares multiple regression (discussed in an earlier learning module) in that it compares the fit of the restricted and unrestricted models; however, it uses the log-likelihoods of each model. The LR test is distributed as chi-squared with q degrees of freedom (i.e., number of restrictions).

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Note the log-likelihood metric is always negative, so higher values (closer to 0) indicate a better-fitting model. Importantly, unlike adjusted R^2 (or $\overline{R}^2$), the log-likelihood metric for a given model is not meaningful by itself but is useful when comparing regression models that have the same dependent variable. Most statistical software produces the log-likelihood metric for the model being estimated, as well as the log-likelihood metric for the intercept-only model (typically designated as LL-Null).

As with the joint *F*-statistic to compare nested models, the null hypothesis for the LR test is that the smaller, restricted model is the better model. For example, if we compare the unrestricted Model A,

$$\ln\left(\frac{P}{1-P}\right) = b_0 + b_1 X_1 + b_2 X_2 + b_3 X_3 + \varepsilon,$$

to Model B, with restrictions $b_2 = b_3 = 0$,

$$\ln\left(\frac{P}{1-P}\right) = b_0 + b_1 X_1 + \varepsilon,$$

then the null hypothesis is H_0: $b_2 = b_3 = 0$ and the alternative hypothesis is that at least one of the coefficients is different from zero.

Here, the LR test is a joint test of the restricted coefficients. Rejecting the null hypothesis is a rejection of the smaller, restricted model in favor of the larger, unrestricted model. Finally, the LR test performs best when applied to large samples.

Logistic regression does not have an equivalent measure to R^2 because it cannot be fitted using least squares. Pseudo-R^2 has been proposed to capture the explained variation in a logistic regression and is generated in standard software output. The pseudo-R^2 must be interpreted with care because it can be used only to compare different specifications of the same model (not models based on different datasets).

KNOWLEDGE CHECK

You are assigned to examine the propensity of companies to repurchase their shares. For a sample of 500 companies, you have identified those that repurchased shares (Repurchase = 1) and those that did not (Repurchase = 0). You also collected company data for the year prior to the repurchase, including cash-to-total-assets ratio (CASH), debt-to-equity ratio (DE), and net profit margin (NPM), and estimated the following logistic regression:

$$\text{Repurchase}_i = b_0 + b_1\text{CASH}_i + b_2\text{DE}_i + b_3\text{NPM}_i + \varepsilon_i.$$

Your regression results are shown in Exhibit 13.

Exhibit 13

Panel A Logistic Regression Results

Dep. Variable	Repurchase	No. Observations	500
Model:	Logit	Df Residuals	496
Method:	MLE	Df Model	3
		Pseudo-R^2	0.0271
		Log-Likelihood	–304.20
Converged:	TRUE	LL-Null	–312.68
Covariance Type:	Non-Robust	LLR-*P*-Value	0.0007

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					95% CI	
	Coefficient	Std. Error	*z*-Stat.	*P*-Value	Lower	Upper
Intercept	−0.4738	0.196	−2.415	0.016	−0.858	−0.089
CASH	−0.9118	1.154	−0.790	0.430	−3.174	1.351
DE	−0.3186	0.133	−2.396	0.017	−0.579	−0.058
NPM	0.9407	0.417	2.255	0.024	0.123	1.758

Panel B Statistics of Independent Variables

Independent Variable	Median	Mean	Standard Deviation
CASH	0.0515	0.083	0.0894
DE	0.8585	0.9182	0.7638
NPM	0.0775	−0.0535	1.4592

In the weekly research team meeting, the research director asks you to explain your logistic regression model and assess how the model fits the data.

1. Interpret the logit regression intercept.

Solution:

The intercept of −0.4738 is the log odds of the probability of being a share repurchaser if CASH, DE, and NPM are all zero. The odds are $e^{-0.4738} = 0.6226$, and the probability (P) = 0.6226/(1 + 0.6226) = 0.3837, or 38.37%. This is the probability not being captured by the independent variables in the logistic regression equation.

2. Estimate the marginal effect of each independent variable in explaining companies' propensity to repurchase shares.

Solution:

Starting with the equation for the probability to repurchase shares,

$$P = \frac{1}{1 + \exp\left[-(b_0 + b_1X_1 + b_2X_2 + b_3X_3)\right]},$$

we use the values of the coefficients from the logistic equation result and the mean, or average values of the independent variables, to find the initial average probability of repurchasing shares:

$$P = \frac{1}{1 + \exp\{-[-0.4738 + (-0.9118)(0.0830) + (0.9407)(-0.0535) + (-0.3186)(0.9182)]\}}.$$

$P = 29.06\%$.

This implies that for the average firm, there is a 29.06% probability of share repurchase.

Now, for each independent variable, let us increase it by 1%, or 0.01, while holding the others constant and see the marginal impact to probability of a share buyback.

CASH:

We increase the CASH variable by 1%, from 0.083 to 0.093, and calculate the new probability of share buyback:

$$P = \frac{1}{1 + \exp\{-[-0.4738 + (-0.9118)(0.0930) + (0.9407)(-0.0535) + (-0.3186)(0.9182)]\}}.$$

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$P = 28.87\%$.

Therefore, the marginal impact of increasing the CASH variable by 1% and holding all the other variables constant is a change in the probability of a share buyback of 28.87% – 29.06% = –0.19%; differently put, increasing the CASH variable by 1% decreases the probability of a buyback by 0.19%.

NPM:

We increase the NPM variable by 1%, from –0.0535 to –0.0435, and calculate the new probability of a share buyback:

$$P = \frac{1}{1 + \exp\{-[-0.4738 + (-0.9118)(0.0830) + (0.9407)(-0.0435) + (-0.3186)(0.9182)]\}}.$$

$P = 29.26\%$

Therefore, the marginal impact of increasing the NPM variable by 1% is an increase in the probability of a share buyback of 29.26% – 29.06% = 0.20%.

DE:

We increase the DE variable by 1%, from 0.9182 to 0.9282, and calculate the new probability of a share buyback:

$$P = \frac{1}{1 + \exp\{-[-0.4738 + (-0.9118)(0.0830) + (0.9407)(-0.0535) + (-0.3186)(0.9282)]\}}.$$

$P = 29.00\%$.

Therefore, the marginal impact of increasing the NPM variable by 1%, rounded to two decimal places, is a decrease in the probability of a share buyback of 29.00% – 29.06% = –0.07%; differently put, it increases the probability of a share buyback.

3. Evaluate how your logistic regression model fits the data using the LR test and an intercept-only model as the restricted model.

Solution:

The log-likelihood statistics from the logistic regression results are as follows:

Model	Log-likelihood
Restricted: Intercept only	–312.68
Unrestricted: Intercept, CASH, DE, NPM	–304.20

The LR test is a test of the hypothesis for the restrictions, using the standard six-step hypothesis test process, as follows:

Step 1	State the hypotheses.	H_0: $b_{1(\text{CASH})} = b_{2(\text{DE})} = b_{3(\text{NPM})} = 0$; H_a: at least one $b_j \neq 0$.
Step 2	Identify the appropriate test statistic.	Likelihood ratio (LR) = –2 (Log-likelihood restricted model – Log-likelihood unrestricted model), and degrees of freedom = 3.
Step 3	Specify the level of significance.	$\alpha = 5\%$.
Step 4	State the decision rule.	Critical chi-square value = 7.8150, where df = 3 and $\alpha = 5\%$, then reject the null hypothesis if the calculated LR is greater than 7.8150.

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Step 5	Calculate the test statistic.	LR = −2(−312.68 − −304.20) = 16.960.
Step 6	Make a decision.	Reject the null hypothesis because the calculated chi-square statistic is greater than the critical chi-square value. We conclude that the unrestricted model fits the data better than the restricted model.

Based on the LR test, your conclusion should be that the unrestricted model fits the data better than the intercept-only model, indicating that the three explanatory variables are jointly significant. Note the regression results show the LR test statistic's *P*-value is 0.0007. Moreover, individual (*z*-statistic) tests of the coefficients show that DE and NPM are each significant at the 5% level.

Code: Logistic Regression

In Python:

```
import pandas as pd
from statsmodels.formula.api import logit
df = pd.read_csv("data.csv")
formula = ('Repurchase ~ CASH + DE + NPM')
model=logit(formula=formula,data=df).
fit(method='newton')
print(model.summary())
```

In R:

```
df <- read.csv('data.csv')
logit <- glm(Repurchase ~ CASH+DE+NPM,
family=binomial(link="logit"),data=df)
summary(logit)
```

Logistic regression plays a key role in binary classification problems in machine learning and neural networks. For example, to enhance fundamental investment analysis, logistic regression can be applied with natural language processing techniques to classify sentiment of financial texts, such as press releases. The dependent variable is sentiment class: positive sentiment (1) and negative sentiment (0), which signal "good" or "bad" corporate news, respectively. The independent variables are tokens (key words or phrases) from financial text, such as company annual reports, earnings releases, and corporate announcements. The logistic regression model is trained to recognize and classify these tokens into good or bad news and is then tested and deployed to enhance investment valuation models.

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PRACTICE PROBLEMS

The following information relates to questions 1-4

The chief investment officer asks you to analyze one of the firm's portfolios to identify influential outliers that might be skewing regression results of its return drivers. For each observation, you calculate leverage and the studentized residual. There are 96 observations and two independent variables ($k = 2$), and the critical t-statistic is 2.63 at a 1% significance level. Partial results of your calculations are shown in Exhibit 1.

Exhibit 1: Regression Data for Detecting Influential Observations

	h_{ii}	Studentized Residual
Observation 1	0.043	2.784
Observation 2	0.022	–0.103
Observation 3	0.036	–0.731
Observation 4	0.059	–0.122
Observation 5	0.011	–0.660
Observation 6	0.101	–2.906
Observation 45	0.042	2.117
Observation 46	0.013	0.172
Observation 47	0.015	–0.672
Observation 48	0.012	–0.734
Observation 49	0.064	0.475
Observation 50	0.141	–2.788
Observation 51	0.011	1.679
Observation 52	0.023	–1.218
Observation 91	0.035	–1.260
Observation 92	0.025	3.001
Observation 93	0.017	1.483
Observation 94	0.097	–0.172
Observation 95	0.017	0.046
Observation 96	0.011	1.819

Finally, you are tasked with investigating whether there is any monthly seasonality in the excess portfolio returns. You construct a regression model using dummy variables for the months; your regression statistics and ANOVA results are shown in Exhibit 2.

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Exhibit 2: Analysis of Monthly Seasonality of Excess Portfolio Returns

Regression Statistics	
Multiple *R*	0.321
R-Squared	0.103
Adjusted *R*-Squared	–0.014
Standard Error	8.100
Observations	96.000

ANOVA

	df	SS	MS	F	Signif. F
Regression	11	634.679	57.698	0.879	0.563
Residual	84	5511.369	65.612		
Total	95	6146.048			

	Coeff.	Std. Error	*t*-Stat.	*P*-Value
Intercept	1.263	2.864	0.441	0.660
Jan	1.311	4.050	0.324	0.747
Feb	–3.756	4.050	–0.927	0.356
Mar	3.495	4.050	0.863	0.391
Apr	0.174	4.050	0.043	0.966
May	0.714	4.050	0.176	0.861
Jun	0.944	4.050	0.233	0.816
Jul	–0.571	4.050	–0.141	0.888
Aug	–0.445	4.050	–0.110	0.913
Sep	–1.744	4.050	–0.431	0.668
Oct	4.261	4.050	1.052	0.296
Nov	–5.311	4.050	–1.311	0.193

1. Determine and justify the potentially influential observation(s) in Exhibit 1 using the leverage measure.

   A. 50, because it has the highest leverage, 0.141

   B. Observations 6 and 50, because their leverage exceeds 0.100

   C. Observations 6, 50, and 94, because their leverage exceeds $3\left(\frac{k+1}{n}\right)$

2. Determine and justify the potentially influential observations in Exhibit 1 using the studentized residuals measure.

   A. Observations 1 and 92, because the values of their studentized residuals exceed 2.63

   B. Observations 1, 6, 50, and 92, because the absolute values of their studentized residuals exceed 2.63

   C. All the observations shown except Observation 95, because the absolute value of its studentized residual is less than 0.094

3. Identify both the base month and the coefficient that represents its returns in

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Exhibit 2.

A. December is the base month, and the intercept coefficient represents its returns.

B. November is the base month, and the intercept coefficient represents its returns.

C. December is the base month, and the average of the coefficients for the other 11 months represents its returns.

4. Determine using Exhibit 2 which one of the following statements is *most likely* to be correct. Monthly seasonality in the firm's portfolio is:

A. highly likely.

B. highly unlikely.

C. not able to be determined from the given data.

The following information relates to questions 5-12

Your second-round interview for the Junior Quantitative Analyst position went well, and the next day, you receive an email from the investment firm congratulating you for making it this far. You are one of four remaining candidates from more than 100 who applied for the position.

Because the position involves quantitative analysis, you are given an assignment to complete within 72 hours. You are provided a dataset and tasked with creating two logistic regression models to predict whether an exchange-traded fund (ETF) will be a "winning" fund—that is, whether the ETF's monthly return will be one standard deviation or more above the mean monthly return across all ETFs in the dataset or whether the ETF will be an "average" fund.

The variables in the dataset are as follows:

	Variable	Description
1	net_assets	Net assets of ETF in US dollars
2	small_fund	Dummy variable = 1 if a small-size fund and 0 otherwise, with large-size fund as the base
3	medium_fund	Dummy variable =1 if a medium-size fund and 0 otherwise, with large-size fund as the base
4	portfolio_stocks	Percentage of portfolio invested in stocks (1 – 100)
5	portfolio_bonds	Percentage of portfolio invested in bonds (1 – 100)
6	price_earnings	Ratio of price per share to earnings per share
7	price_book	Ratio of price per share to book value per share
8	price_sales	Ratio of price per share to sales per share
9	price_cashflow	Ratio of price per share to cash flow per share
10	label (dependent variable)	1 = winning fund, an ETF whose monthly return is one standard deviation or more above the mean monthly return across all ETFs in the dataset, and 0 = average fund, otherwise

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For the first logistic regression, you are asked to use all the independent variables, except for the fund size dummy variables (small_fund and medium_fund). For the second logistic regression, you are asked to use all the independent variables except the fund size continuous variable (net assets).

You use a standard software package (in Python or R) to develop the logistic regression models. Your results are as follows:

Logistic Regression 1

Logistic Regression Results

Model: Logit		Pseudo-R^2:	0.057	
Dependent Variable:	label			
No. Observations:	1,594	Log-Likelihood:	−451.66	
Df Model:	7	LL-Null:	−478.86	
Df Residuals:	1,586	LLR	*P*-Value:	1.97E−9
No. Iterations:	10			

	Coefficient	Std. Err.	*z*-Statistic	*P*-Value > |*z*|	[0.025	0.975]
Intercept	−2.0350	0.5221	−3.8979	0.0001	−3.0583	−1.0118
net_assets	−0.7667	1.3571	−0.5649	0.5721	−3.4265	1.8932
portfolio_stocks	−0.0089	0.0051	−1.7550	0.0793	−0.0188	0.0010
portfolio_bonds	−0.1113	0.0729	−1.5263	0.1269	−0.2543	0.0316
price_earnings	0.0292	0.0200	1.4647	0.1430	−0.0099	0.0683
price_book	−0.0390	0.1029	−0.3791	0.7046	−0.2407	0.1627
price_sales	0.3432	0.0777	4.4160	0.0000	0.1909	0.4956
price_cashflow	−0.0502	0.0363	−1.3805	0.1674	−0.1214	0.0211

Logistic Regression 2

Logistic Regression Results

Model: Logit		Pseudo-R^2:	0.059	
Dependent Variable:	label			
No. Observations:	1,594	Log-Likelihood:	−450.4	
Df Model:	8	LL-Null:	−478.86	
Df Residuals:	1,585	LLR	*P*-Value:	1.87E−09
No. Iterations:	10			

	Coefficient	Std. Err.	*z*-Statistic	*P*-Value > |*z*|	[0.025	0.975]
Intercept	−1.9589	0.5254	−3.7283	0.0002	−2.9886	−0.9291
small_fund	−0.4794	0.3719	−1.2888	0.1975	−1.2083	0.2496
medium_fund	−0.3509	0.2348	−1.4948	0.1350	−0.8111	0.1092
portfolio_stocks	−0.0092	0.0051	−1.8099	0.0703	−0.0191	0.0008

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	Coefficient	Std. Err.	*z*-Statistic	*P*-Value > |*z*|	[0.025	0.975]
portfolio_bonds	–0.1121	0.0727	–1.5433	0.1228	–0.2546	0.0303
price_earnings	0.0389	0.0211	1.8467	0.0648	–0.0024	0.0802
price_book	–0.0803	0.1068	–0.7519	0.4521	–0.2897	0.1291
price_sales	0.3453	0.0796	4.3362	0.0000	0.1892	0.5014
price_cashflow	–0.0510	0.0376	–1.3573	0.1747	–0.1247	0.0227

5. Identify which one of the following choices is *most likely* to be correct: Logistic regression is the appropriate regression method for your assignment because the:

   A. independent variables include dummy variables for small- and medium-size funds.

   B. dependent variable is binary rather than continuous.

   C. dependent variable is not continuous and the independent variables include dummy variables for small- and medium-size funds.

6. Identify which one of the following statements *best describes* the interpretation of an independent variable's slope coefficient in a logistic regression model: The slope coefficient is the change in the:

   A. log odds that the event happens per unit change in the independent variable, while all other independent variables increase by one unit.

   B. odds that the event happens per unit change in the independent variable, holding all other independent variables constant.

   C. log odds that the event happens per unit change in the independent variable, holding all other independent variables constant.

7. Determine which one of the following statements is *true*. The intercept in these logistic regressions is interpreted as the:

   A. probability of the ETF being a winning fund if all independent variables are one.

   B. log odds of the ETF being a winning fund if all independent variables are zero.

   C. log odds of the ETF being an average fund if all independent variables are zero.

8. Based on the output for Logistic Regression 1 in the table below, which of the following alternatives is closest to the probability that any ETF will be a winning fund?

   A. 6.75%

   B. 5.96%

   C. 5.67%

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Independent Variables	Median	Mean	Standard Deviation
net_assets ($100B)	0.1687	0.2911	3.5255
portfolio_stocks	99.6400	92.9093	19.6162
portfolio_bonds	0	2.3068	11.9835
price_earnings	14.4750	15.1743	5.4087
price_book	1.7800	2.0711	1.1113
price_sales	1.2600	1.6060	1.1877
price_cashflow	7.0300	7.6489	3.7768

9. Based on the output for Logistic Regression 1 and the information in the table above, which of the following alternatives is closest to the change in the probability that an ETF will be a winning fund if its price-to-earnings ratio increases by one unit and all else stays constant?

A. 17.0%

B. 0.16%

C. 1.5%

10. Based on the output from with Logistic Regression 1, how will the change in the probability that an ETF will be a winning fund increase if one of the other independent variable values, except for net_assets, is decreased by one unit, holding all else constant?

A. The probability will increase, but not as much as with the price-to-earnings increasing by one unit.

B. The probability will increase more than the price-to-earnings increasing by one unit.

C. The probability will not increase.

11. Identify which one of the following statements about the logistic regression model fit is *most likely* to be correct: Based on the log-likelihood criteria:

A. Model 2 has a better fit because it has a higher log-likelihood value.

B. Model 2 has a better fit because it has a lower log-likelihood value.

C. Model 1 has a better fit because it has a higher log-likelihood value.

12. Determine, using only the statistically significant variable estimate(s) in Logistic Regression 2 and the information provided below, which of the following is *closest to* the probability of the Alpha ETF being a winning fund and whether it would be classified as a winning fund.

Alpha ETF variable values:

- small_fund = 0.
- medium_fund = 0.
- portfolio_stocks = 99.3%.
- portfolio_bonds = 0.7%.
- price_earnings = 25.0.
- price_book =1.1.

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- price_sales = 4.0.
- price_cashflow = 5.7.

Use a significance level of 5% and a probability threshold for being a winner of 65%.

A. 27.4%, and the Alpha ETF is not classified as a winning fund

B. 36.0%, and the Alpha ETF is not classified as a winning fund

C. 82.2%, and the Alpha ETF is classified as a winning fund

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SOLUTIONS

1. C is correct. The rule of thumb for the leverage measure is that if it exceeds $3\left(\frac{k+1}{n}\right)$, where k is the number of independent variables, then it is a potentially influential observation. Since $n = 96$ and $k = 2$, then $3 = \left(\frac{2+1}{96}\right) = 0.09375$.

   Three observations exceed this value: 6, 50, and 94. So, they are potentially influential observations.

2. B is correct. For the studentized residuals measure, the critical t-value is 2.63. So, any observation with a studentized residual whose absolute value exceeds 2.63 is a potentially influential observation. The studentized residuals for Observations 1, 6, 50, and 92 have absolute values exceeding 2.63; therefore, they are potentially influential observations.

3. A is correct. December is the base month, and the intercept coefficient represents its returns. We use 11 dummy variables to represent the returns for each month from January through November. December results are measured when each of these dummy variables equals zero, leaving the intercept coefficient to represent December returns.

4. B is correct. Monthly seasonality in the firm's portfolio is highly unlikely. The variance explained by the model (R-squared) is only 10.3%, and after adjusting for the number of independent variables (adjusted R-squared), it becomes negative. Also, the insignificant F-statistic indicates a 56.3% chance that all variable coefficients are zero. Finally, t-statistics and associated p-values indicate that all the variable coefficients are insignificant (i.e., not significantly different from zero). Consequently, monthly seasonality is highly unlikely to exist in this portfolio.

5. B is correct. Logistic regression is the appropriate regression method because the dependent variable is binary rather than continuous.

6. C is correct. An independent variable's slope coefficient in a logistic regression model is the change in the log odds that the event happens per unit change in the independent variable, holding all other independent variables constant.

7. B is correct. The intercept in these logistic regressions is interpreted as the log odds of the ETF being a winning fund if all independent variables are zero.

8. C is correct. We calculate the probability that an that an ETF will be a winning fund by using the variable estimates and the average values of the independent variables.

   Using the equation for the probability, where we have seven independent variables,

   $$P = \frac{1}{1 + \exp[-b_0 + b_1 X_1 + b_2 X_2 + \ldots b_7 X_7]}.$$

   Using the mean values and coefficient estimates of the independent variables, the probability of the average ETF being a winner is

   $$P = \frac{1}{1 + \exp\left\{-\left[\begin{array}{l}-2.0350 + (-0.7667)(0.2911) + (-0.0089)(92.9093) + \\ (-0.1113)(2.3068) + (0.0292)(15.1743) + (0.0390)(2.0711) + \\ (0.3432)(1.6060) + (-0.0502)(7.6489)\end{array}\right]\right\}}$$

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$P = 5.67\%$.

This implies that for an ETF with the average values of the independent variables, there is a 5.67% probability that it will be a winning ETF.

9. B is correct. Keeping all the independent variables' average values fixed and increasing the P/E value by 1, from 15.1743 to 16.1743, yields the new probability.

$$P = \frac{1}{1 + \exp\left\{ - \begin{bmatrix} -2.0350 + (-0.7667)(0.2911) + (-0.0089)(92.9093) + \\ (-0.1113)(2.3068) + (0.0292)(16.1743) + (-0.0390)(2.0711) + \\ (0.3432)(1.6060) + (-0.0502)(7.6489) \end{bmatrix} \right\}}$$

$P = 5.82\%$.

From the previous question, we know that the marginal probability is 5.67%. Therefore, the marginal increase in the probability of an average ETF being a winner went up 5.82% – 5.67% = 0.15%. A one-unit change in the P/E of an ETF will increase the probability of it becoming a winning fund by 0.15%, and therefore, B is the closest answer.

10. B is correct. In the previous question, the price-to-earnings variable value and the coefficient are both positive. By increasing the variable value incrementally by one, we are increasing the overall positive value of the series of items in the exp function. Therefore, if we are reducing the product of a coefficient value pair that is negative, we are increasing the overall value of the series of items in the exp function.

The next step is to look to see how many negative coefficient and value products are in the series of items in the exp function, then calculate the coefficient value product, and compare them to the coefficient value product for the price-to-earnings variable.

Variable	Coeff.	Avg. Value	Initial Product	Avg. Value Change	Avg. Value + Change	New Product	Increase
portfolio_ stocks coeff	–0.0089	92.9093	–0.8269	–1	91.9093	–0.8180	0.0089
portfolio_ bonds coeff	–0.1113	2.3068	–0.2567	–1	1.3068	–0.1454	0.1113
price_earnings coeff	0.0292	15.1743	0.4431	1	16.1743	0.4723	0.0292
price_book coeff	–0.0390	2.0711	–0.0808	–1	1.0711	–0.0418	0.0390
price_cashflow coeff	–0.0502	7.6489	–0.3840	–1	6.6489	–0.3338	0.0502

Therefore, as the portfolio_bonds variable increases by one unit, it results in a larger increase in profit than the price-to-earnings variable (0.1113 versus 0.0292), since its product is larger than the price-to-earnings product increase by one unit.

11. A is correct. Model 2 has a better fit because it has a higher (less negative) log-likelihood value, –450.40 versus –451.66, compared to Model 1.

12. B is correct. Besides the significant intercept, the only significant (at the 5% level) variable in Logistic Regression 2 is price_sales. Using these two factors, the probability of this ETF being a winning fund is calculated to be 35.95%, as follows:

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$$\text{Probability of being a winning fund} = 0.3595 = \frac{1}{1 + \exp[-(-1.9589 + (0.3453)(4.0))]}.$$

Because this probability is well below the 65% threshold for being a winner, the Alpha ETF would not be classified as a winning fund.

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LEARNING MODULE

5

Time-Series Analysis

by Richard A. DeFusco, PhD, CFA, Dennis W. McLeavey, DBA, CFA, Jerald E. Pinto, PhD, CFA, and David E. Runkle, PhD, CFA.

Richard A. DeFusco, PhD, CFA, is at the University of Nebraska-Lincoln (USA). Dennis W. McLeavey, DBA, CFA, is at the University of Rhode Island (USA). Jerald E. Pinto, PhD, CFA, is at CFA Institute (USA). David E. Runkle, PhD, CFA, is at Jacobs Levy Equity Management (USA).

LEARNING OUTCOMES

Mastery	*The candidate should be able to:*
☐	calculate and evaluate the predicted trend value for a time series, modeled as either a linear trend or a log-linear trend, given the estimated trend coefficients
☐	describe factors that determine whether a linear or a log-linear trend should be used with a particular time series and evaluate limitations of trend models
☐	explain the requirement for a time series to be covariance stationary and describe the significance of a series that is not stationary
☐	describe the structure of an autoregressive (AR) model of order p and calculate one- and two-period-ahead forecasts given the estimated coefficients
☐	explain how autocorrelations of the residuals can be used to test whether the autoregressive model fits the time series
☐	explain mean reversion and calculate a mean-reverting level
☐	contrast in-sample and out-of-sample forecasts and compare the forecasting accuracy of different time-series models based on the root mean squared error criterion
☐	explain the instability of coefficients of time-series models
☐	describe characteristics of random walk processes and contrast them to covariance stationary processes
☐	describe implications of unit roots for time-series analysis, explain when unit roots are likely to occur and how to test for them, and demonstrate how a time series with a unit root can be transformed so it can be analyzed with an AR model
☐	describe the steps of the unit root test for nonstationarity and explain the relation of the test to autoregressive time-series models

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LEARNING OUTCOMES

Mastery	*The candidate should be able to:*
☐	explain how to test and correct for seasonality in a time-series model and calculate and interpret a forecasted value using an AR model with a seasonal lag
☐	explain autoregressive conditional heteroskedasticity (ARCH) and describe how ARCH models can be applied to predict the variance of a time series
☐	explain how time-series variables should be analyzed for nonstationarity and/or cointegration before use in a linear regression
☐	determine an appropriate time-series model to analyze a given investment problem and justify that choice

1 INTRODUCTION

As financial analysts, we often use time-series data to make investment decisions. A **time series** is a set of observations on a variable's outcomes in different time periods: the quarterly sales for a particular company during the past five years, for example, or the daily returns on a traded security. In this reading, we explore the two chief uses of time-series models: to explain the past and to predict the future of a time series. We also discuss how to estimate time-series models, and we examine how a model describing a particular time series can change over time. The following two examples illustrate the kinds of questions we might want to ask about time series.

Suppose it is the beginning of 2020 and we are managing a US-based investment portfolio that includes Swiss stocks. Because the value of this portfolio would decrease if the Swiss franc depreciates with respect to the dollar, and vice versa, holding all else constant, we are considering whether to hedge the portfolio's exposure to changes in the value of the franc. To help us in making this decision, we decide to model the time series of the franc/dollar exchange rate. Exhibit 1 shows monthly data on the franc/dollar exchange rate. The data are monthly averages of daily exchange rates. Has the exchange rate been more stable since 1987 than it was in previous years? Has the exchange rate shown a long-term trend? How can we best use past exchange rates to predict future exchange rates?

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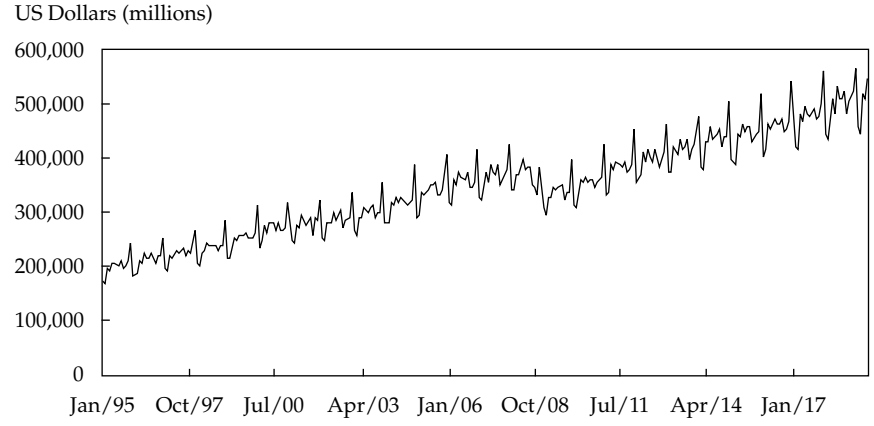
Exhibit 1: Swiss Franc/US Dollar Exchange Rate, Monthly Average of Daily Data

Source: Board of Governors of the Federal Reserve System.

As another example, suppose it is the beginning of 2020. We cover retail stores for a sell-side firm and want to predict retail sales for the coming year. Exhibit 2 shows monthly data on US retail sales. The data are not seasonally adjusted, hence the spikes around the holiday season at the turn of each year. Because the reported sales in the stores' financial statements are not seasonally adjusted, we model seasonally unadjusted retail sales. How can we model the trend in retail sales? How can we adjust for the extreme seasonality reflected in the peaks and troughs occurring at regular intervals? How can we best use past retail sales to predict future retail sales?

Exhibit 2: Monthly US Retail Sales



Source: US Department of Commerce, Census Bureau.

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Some fundamental questions arise in time-series analysis: How do we model trends? How do we predict the future value of a time series based on its past values? How do we model seasonality? How do we choose among time-series models? And how do we model changes in the variance of time series over time? We address each of these issues in this reading.

We first describe typical challenges in applying the linear regression model to time-series data. We present linear and log-linear trend models, which describe, respectively, the value and the natural log of the value of a time series as a linear function of time. We then present autoregressive time-series models—which explain the current value of a time series in terms of one or more lagged values of the series. Such models are among the most commonly used in investments, and the section addresses many related concepts and issues. We then turn our attention to random walks. Because such time series are not covariance stationary, they cannot be modeled using autoregressive models unless they can be transformed into stationary series. We therefore explore appropriate transformations and tests of stationarity. The subsequent sections address moving-average time-series models and discuss the problem of seasonality in time series and how to address it. We also cover autoregressive moving-average models, a more complex alternative to autoregressive models. The last two topics are modeling changing variance of the error term in a time series and the consequences of regression of one time series on another when one or both time series may not be covariance stationary.

Challenges of Working with Time Series

Throughout the reading, our objective will be to apply linear regression to a given time series. Unfortunately, in working with time series, we often find that the assumptions of the linear regression model are not satisfied. To apply time-series analysis, we need to assure ourselves that the linear regression model assumptions are met. When those assumptions are not satisfied, in many cases we can transform the time series or specify the regression model differently, so that the assumptions of the linear regression model are met.

We can illustrate assumption difficulties in the context of a common time-series model, an autoregressive model. Informally, an autoregressive model is one in which the independent variable is a lagged (that is, past) value of the dependent variable, such as the model $x_t = b_0 + b_1 x_{t-1} + \varepsilon_t$ (we could also write the equation as $y_t = b_0 + b_1 y_{t-1} + \varepsilon_t$). Specific problems that we often encounter in dealing with time series include the following:

- The residual errors are correlated instead of being uncorrelated. In the calculated regression, the difference between x_t and $b_0 + b_1 x_{t-1}$ is called the residual error (ε_t). The linear regression assumes that this error term is not correlated across observations. The violation of that assumption is frequently more critical in terms of its consequences in the case of time-series models involving past values of the time series as independent variables than for other models (such as cross-sectional models) in which the dependent and independent variables are distinct. As we discussed in the reading on multiple regression, in a regression in which the dependent and independent variables are distinct, serial correlation of the errors in this model does not affect the consistency of our estimates of intercept or slope coefficients. By contrast, in an autoregressive time-series regression, such as $x_t = b_0 + b_1 x_{t-1} + \varepsilon_t$, serial correlation in the error term causes estimates of the intercept (b_0) and slope coefficient (b_1) to be inconsistent.

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- The mean or variance of the time series changes over time. Regression results are invalid if we estimate an autoregressive model for a time series with mean or variance that changes over time.

Before we try to use time series for forecasting, we may need to transform the time-series model so that it is well specified for linear regression. With this objective in mind, you will observe that time-series analysis is relatively straightforward and logical.

LINEAR TREND MODELS

2

☐ calculate and evaluate the predicted trend value for a time series, modeled as either a linear trend or a log-linear trend, given the estimated trend coefficients

Estimating a trend in a time series and using that trend to predict future values of the time series is the simplest method of forecasting. For example, we saw in Exhibit 2 that monthly US retail sales show a long-term pattern of upward movement—that is, a **trend**. In this section, we examine two types of trends—linear trends and log-linear trends—and discuss how to choose between them.

Linear Trend Models

The simplest type of trend is a **linear trend**, one in which the dependent variable changes at a constant rate with time. If a time series, y_t, has a linear trend, then we can model the series using the following regression equation:

$$y_t = b_0 + b_1 t + \varepsilon_t,\ t = 1, 2, \ldots, T, \tag{1}$$

where

y_t = the value of the time series at time t (value of the dependent variable)

b_0 = the y-intercept term

b_1 = the slope coefficient

t = time, the independent or explanatory variable

ε_t = a random error term

In Equation 1, the trend line, $b_0 + b_1 t$, predicts the value of the time series at time t (where t takes on a value of 1 in the first period of the sample and increases by 1 in each subsequent period). Because the coefficient b_1 is the slope of the trend line, we refer to b_1 as the trend coefficient. We can estimate the two coefficients, b_0 and b_1, using ordinary least squares, denoting the estimated coefficients as $\hat{b}_0$ and $\hat{b}_1$. Recall that ordinary least squares is an estimation method based on the criterion of minimizing the sum of a regression's squared residuals.

Now we demonstrate how to use these estimates to predict the value of the time series in a particular period. Recall that t takes on a value of 1 in Period 1. Therefore, the predicted or fitted value of y_t in Period 1 is $\hat{y}_1 = \hat{b}_0 + \hat{b}_1(1)$. Similarly, in a subsequent period—say, the sixth period—the fitted value is $\hat{y}_6 = \hat{b}_0 + \hat{b}_1(6)$. Now suppose that we want to predict the value of the time series for a period outside the sample—say, period $T + 1$. The predicted value of y_t for period $T + 1$ is $\hat{y}_{T+1} = \hat{b}_0 + \hat{b}_1(T + 1)$. For example, if $\hat{b}_0$ is 5.1 and $\hat{b}_1$ is 2, then at $t = 5$ the pre-

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dicted value of y_5 is 15.1 and at $t = 6$ the predicted value of y_6 is 17.1. Note that each consecutive observation in this time series increases by $\hat{b}_1 = 2$, irrespective of the level of the series in the previous period.

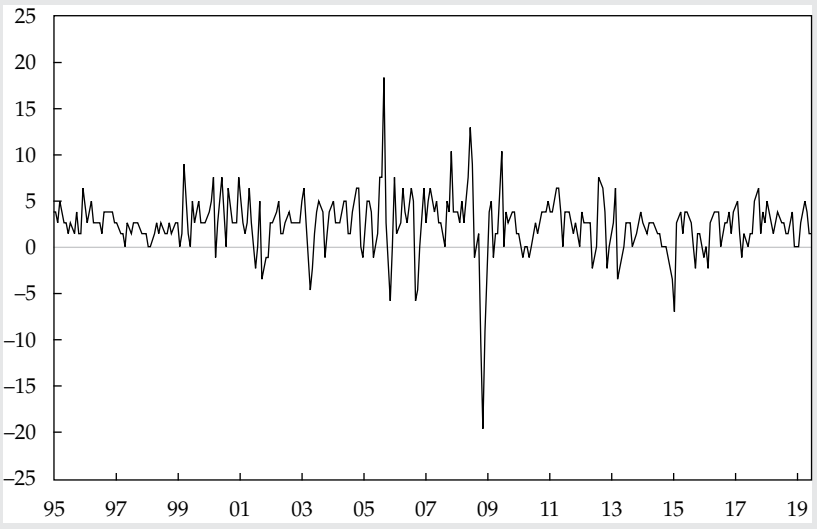
EXAMPLE 1

The Trend in the US Consumer Price Index

It is January 2020. As a fixed-income analyst in the trust department of a bank, Lisette Miller is concerned about the future level of inflation and how it might affect portfolio value. Therefore, she wants to predict future inflation rates. For this purpose, she first needs to estimate the linear trend in inflation. To do so, she uses the monthly US Consumer Price Index (CPI) inflation data, expressed as an annual percentage rate, (1% is represented as 1.0) shown in Exhibit 3. The data include 228 months from January 1995 through June 2019, and the model to be estimated is $y_t = b_0 + b_1 t + \varepsilon_t$, $t = 1, 2, \ldots, 294$. The table in Exhibit 4 shows the results of estimating this equation. With 294 observations and two parameters, this model has 292 degrees of freedom. At the 0.05 significance level, the critical value for a *t*-statistic is 1.97. The intercept $\left(\hat{b}_0 = 2.7845\right)$ is statistically significant because the value of the *t*-statistic for the coefficient is well above the critical value. The trend coefficient is negative $\left(\hat{b}_1 = -0.0037\right)$, suggesting a slightly declining trend in inflation during the sample time period. However, the trend is not statistically significant because the absolute value of the *t*-statistic for the coefficient is below the critical value. The estimated regression equation can be written as

$$y_t = 2.7845 - 0.0037t.$$

Exhibit 3: Monthly CPI Inflation, Not Seasonally Adjusted



Source: Bureau of Labor Statistics.

Exhibit 4: Estimating a Linear Trend in Inflation: Monthly Observations, January 1995–June 2019

Regression Statistics	
R^2	0.0099
Standard error	3.1912

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Regression Statistics	
Observations	294
Durbin–Watson	1.2145

	Coefficient	Standard Error	*t*-Statistic
Intercept	2.7845	0.3732	7.4611
t (Trend)	−0.0037	0.0022	−1.68

Source: US Bureau of Labor Statistics.

Because the trend line slope is estimated to be −0.0037, Miller concludes that the linear trend model's best estimate is that the annualized rate of inflation declined at a rate of about 37 bps per month during the sample time period. The decline is not statistically significantly different from zero.

In January 1995, the first month of the sample, the predicted value of inflation is $\hat{y}_1 = 2.7845 - 0.0037(1) = 2.7808\%$. In June 2019, the 294th, or last, month of the sample, the predicted value of inflation is $\hat{y}_{228} = 2.7845 - 0.0037(294) = 1.697\%$. Note, though, that these predicted values are for in-sample periods. A comparison of these values with the actual values indicates how well Miller's model fits the data; however, a main purpose of the estimated model is to predict the level of inflation for out-of-sample periods. For example, for June 2020 (12 months after the end of the sample), $t = 294 + 12 = 306$, and the predicted level of inflation is $\hat{y}_{306} = 2.7845 - 0.0037(306) = 1.6523\%$.

Exhibit 5 shows the inflation data along with the fitted trend. Consistent with the negative but small and statistically insignificant trend coefficient, the fitted trend line is slightly downward sloping. Note that inflation does not appear to be above or below the trend line for a long period of time. No persistent differences exist between the trend and actual inflation. The residuals (actual minus trend values) appear to be unpredictable and uncorrelated in time. Therefore, using a linear trend line to model inflation rates from 1995 through 2019 does not appear to violate the assumptions of the linear regression model. Note also that the R^2 in this model is quite low, indicating great uncertainty in the inflation forecasts from this model. In fact, the estimated model explains only 0.99% of the variation in monthly inflation. Although linear trend models have their uses, they are often inappropriate for economic data. Most economic time series reflect trends with changing slopes and/or intercepts over time. The linear trend model identifies the slope and intercept that provides the best linear fit for all past data. The model's deviation from the actual data can be greatest near the end of a data series, which can compromise forecasting accuracy. Later in this reading, we will examine whether we can build a better model of inflation than a model that uses only a trend line.

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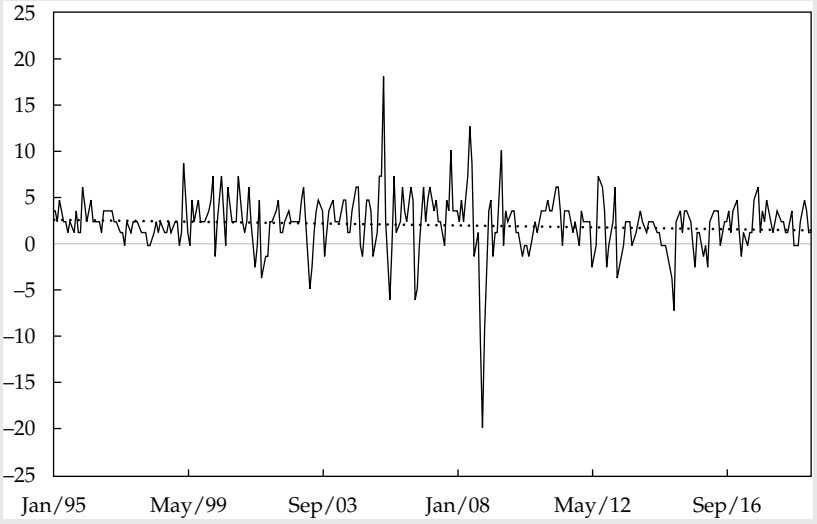
Exhibit 5: Monthly CPI Inflation with Trend



Source: US Bureau of Labor Statistics.

3 LOG-LINEAR TREND MODELS

- ☐ calculate and evaluate the predicted trend value for a time series, modeled as either a linear trend or a log-linear trend, given the estimated trend coefficients
- ☐ describe factors that determine whether a linear or a log-linear trend should be used with a particular time series and evaluate limitations of trend models

Sometimes a linear trend does not correctly model the growth of a time series. In those cases, we often find that fitting a linear trend to a time series leads to persistent rather than uncorrelated errors. If the residuals from a linear trend model are persistent, then we need to employ an alternative model satisfying the conditions of linear regression. For financial time series, an important alternative to a linear trend is a log-linear trend. Log-linear trends work well in fitting time series that have exponential growth.

Exponential growth means constant growth at a particular rate. For example, annual growth at a constant rate of 5% is exponential growth. How does exponential growth work? Suppose we describe a time series by the following equation:

$$y_t = e^{b_0+b_1 t}, \quad t = 1, \quad 2, \quad \ldots, \quad T. \tag{2}$$

Exponential growth is growth at a constant rate $\left(e^{b_1} - 1\right)$ with continuous compounding. For instance, consider values of the time series in two consecutive periods. In Period 1, the time series has the value $y_1 = e^{b_0+b_1(1)}$, and in Period 2, it has the value $y_2 = e^{b_0+b_1(2)}$. The resulting ratio of the values of the time series in the first two periods is $y_2/y_1 = \left(e^{b_0+b_1(2)}\right) / \left(e^{b_0+b_1(1)}\right) = e^{b_1(1)}$. Generally, in any period t, the time series has the value $y_t = e^{b_0+b_1(t)}$. In period $t + 1$, the time series has the value $y_{t+1} = e^{b_0+b_1(t+1)}$. The ratio of the values in the periods $(t + 1)$ and t is $y_{t+1}/y_t =$

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$e^{b_0+b_1(t+1)}/e^{b_0+b_1(t)} = e^{b_1(1)}$. Thus, the proportional rate of growth in the time series over two consecutive periods is always the same: $(y_{t+1} - y_t)/y_t = y_{t+1}/y_t - 1 = e^{b_1} - 1$. For example, if we use annual periods and $e^{b_1} = 1.04$ for a particular series, then that series grows by 1.04 – 1 = 0.04, or 4% a year. Therefore, exponential growth is growth at a constant rate. Continuous compounding is a mathematical convenience that allows us to restate the equation in a form that is easy to estimate.

If we take the natural log of both sides of Equation 2, the result is the following equation:

$$\ln y_t = b_0 + b_1 t,\ t = 1, 2, \ldots, T.$$

Therefore, if a time series grows at an exponential rate, we can model the natural log of that series using a linear trend (an exponential growth rate is a compound growth rate with continuous compounding). Of course, no time series grows exactly at a constant rate. Consequently, if we want to use a **log-linear model**, we must estimate the following equation:

$$\ln y_t = b_0 + b_1 t + \varepsilon_t,\ t = 1, 2, \ldots, T. \quad (3)$$

Note that this equation is linear in the coefficients b_0 and b_1. In contrast to a linear trend model, in which the predicted trend value of y_t is $\hat{b}_0 + \hat{b}_1 t$, the predicted trend value of y_t in a log-linear trend model is $e^{\hat{b}_0+\hat{b}_1 t}$ because $e^{\ln y_t} = y_t$.

Examining Equation 3, we see that a log-linear model predicts that $\ln y_t$ will increase by b_1 from one time period to the next. The model predicts a constant growth rate in y_t of $e^{b_1} - 1$. For example, if $b_1 = 0.05$, then the predicted growth rate of y_t in each period is $e^{0.05} - 1 = 0.051271$, or 5.13%. In contrast, the linear trend model (Equation 1) predicts that y_t grows by a constant amount from one period to the next.

Example 2 illustrates the problem of nonrandom residuals in a linear trend model, and Example 3 shows a log-linear regression fit to the same data.

EXAMPLE 2

A Linear Trend Regression for Quarterly Sales at Starbucks

In September 2019, technology analyst Ray Benedict wants to use Equation 1 to fit the data on quarterly sales for Starbucks Corporation shown in Exhibit 6. Starbucks' fiscal year ends in June. Benedict uses 74 observations on Starbucks' sales from the second quarter of fiscal year 2001 (starting in April 2001) to the third quarter of fiscal year 2019 (ending in June 2019) to estimate the linear trend regression model $y_t = b_0 + b_1 t + \varepsilon_t$, $t = 1, 2, \ldots, 74$. Exhibit 7 shows the results of estimating this equation.

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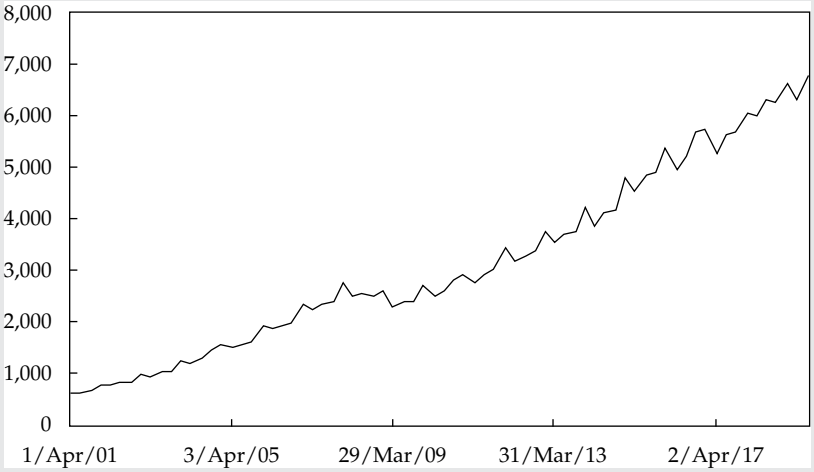
Exhibit 6: Starbucks Quarterly Sales by Fiscal Year



Source: Bloomberg.

Exhibit 7: Estimating a Linear Trend in Starbucks Sales

Regression Statistics	
R^2	0.9603
Standard error	353.36
Observations	74
Durbin–Watson	0.40

	Coefficient	Standard Error	*t*-Statistic
Intercept	137.4213	82.99	1.6559
t (Trend)	80.2060	1.9231	41.7066

Source: Bloomberg.

At first glance, the results shown in Exhibit 7 seem quite reasonable: The trend coefficient is highly statistically significant. When Benedict plots the data on Starbucks' sales and the trend line, however, he sees a different picture. As Exhibit 8 shows, before 2008 the trend line is persistently below sales. Subsequently, until 2015, the trend line is persistently above sales and then varies somewhat thereafter.

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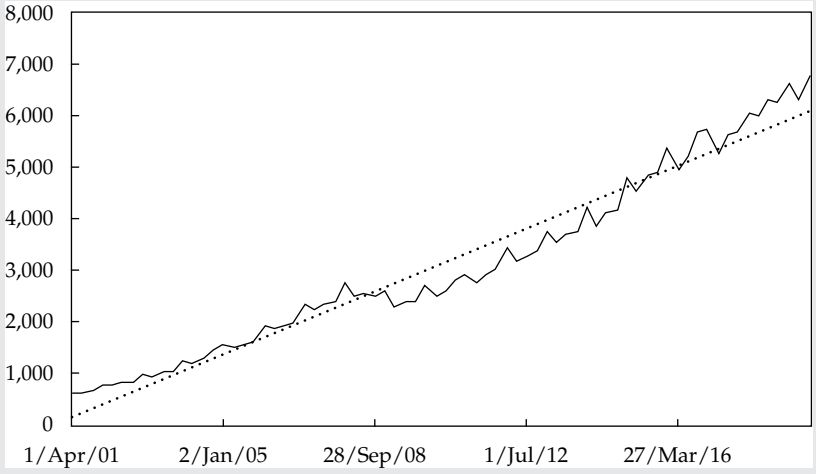
Exhibit 8: Starbucks Quarterly Sales with Trend

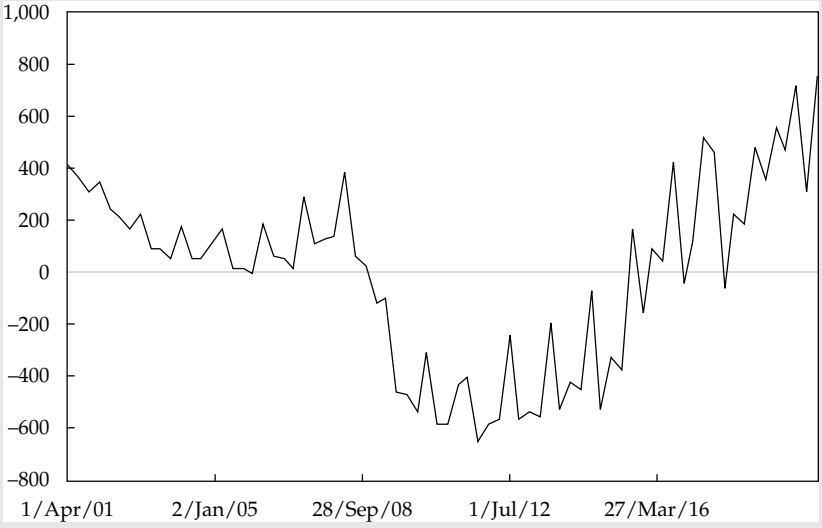


Source: Bloomberg.

Recall a key assumption underlying the regression model: that the regression errors are not correlated across observations. If a trend is persistently above or below the value of the time series, however, the residuals (the difference between the time series and the trend) are serially correlated. Exhibit 9 shows the residuals (the difference between sales and the trend) from estimating a linear trend model with the raw sales data. The figure shows that the residuals are persistent: They are consistently negative from 2008 to 2015 and consistently positive from 2001 to 2008 and from 2017 to 2019.

Because of this persistent serial correlation in the errors of the trend model, using a linear trend to fit sales at Starbucks would be inappropriate, even though the R^2 of the equation is high (0.96). The assumption of uncorrelated residual errors has been violated. Because the dependent and independent variables are not distinct, as in cross-sectional regressions, this assumption violation is serious and causes us to search for a better model.

Exhibit 9: Residual from Predicting Starbucks Sales with a Trend



Source: Bloomberg.

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EXAMPLE 3

A Log-Linear Regression for Quarterly Sales at Starbucks

1. Having rejected a linear trend model in Example 2, technology analyst Benedict now tries a different model for the quarterly sales for Starbucks Corporation from the second quarter of 2001 to the third quarter of 2019. The curvature in the data plot shown in Exhibit 6 provides a hint that an exponential curve may fit the data. Consequently, he estimates the following linear equation:

$$\ln y_t = b_0 + b_1 t + \varepsilon_t,\ t = 1, 2, \ldots, 74.$$

This equation seems to fit the sales data well. As Exhibit 10 shows, the R^2 for this equation is 0.95. An R^2 of 0.95 means that 95% of the variation in the natural log of Starbucks' sales is explained solely by a linear trend.

Exhibit 10: Estimating a Linear Trend in Lognormal Starbucks Sales

Regression Statistics	
R^2	0.9771
Standard error	0.1393
Observations	74
Durbin–Watson	0.26

	Coefficient	Standard Error	*t*-Statistic
Intercept	6.7617	0.0327	206.80
t (Trend)	0.0295	0.0008	36.875

Source: Compustat.

Although both Equations 1 and Equation 3 have a high R^2, Exhibit 11 shows how well a linear trend fits the natural log of Starbucks' sales (Equation 3). The natural logs of the sales data lie very close to the linear trend during the sample period, and log sales are not substantially above or below the trend for long periods of time. Thus, a log-linear trend model seems better suited for modeling Starbucks' sales than a linear trend model is.

1. Benedict wants to use the results of estimating Equation 3 to predict Starbucks' sales in the future. What is the predicted value of Starbucks' sales for the fourth quarter of 2019?

Solution:

The estimated value $\hat{b}_0$ is 6.7617, and the estimated value $\hat{b}_1$ is 0.0295.

Therefore, for fourth quarter of 2019 (t = 75), the estimated model predicts that $\ln \hat{y}_{75} = 6.7617 + 0.0295(75) = 8.9742$ and that sales will be $\hat{y} = e^{\ln \hat{y}_{75}} = e^{8.9742} = \$7{,}896.7$ million. Note that a $\hat{b}_1$ of 0.0295 implies that the exponential growth rate per quarter in Starbucks' sales will be 2.99394% ($e^{0.0295} - 1 = 0.0299394$).

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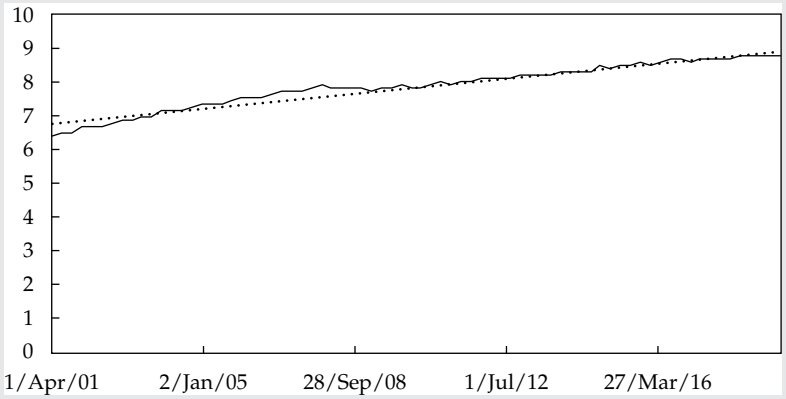
Exhibit 11: Natural Log of Starbucks Quarterly Sales



Source: Compustat.

2. How much different is the previous forecast from the prediction of the linear trend model?

Solution:

Exhibit 7 showed that for the linear trend model, the estimated value of $\hat{b}_0$ is 137.4213 and the estimated value of $\hat{b}_1$ is 80.2060. Thus, if we predict Starbucks' sales for the fourth quarter of 2019 (t = 75) using the linear trend model, the forecast is $\hat{y}_{75}$ = 137.4213 + 80.2060(75) = $6,152.87 million. This forecast is far below the prediction made by the log-linear regression model. Later we will examine whether we can build a better model of Starbucks' quarterly sales than a model that uses only a log-linear trend.

TREND MODELS AND TESTING FOR CORRELATED ERRORS

4

☐ describe factors that determine whether a linear or a log-linear trend should be used with a particular time series and evaluate limitations of trend models

Both the linear trend model and the log-linear trend model are single-variable regression models. If they are to be correctly specified, the regression model assumptions must be satisfied. In particular, the regression error for one period must be uncorrelated with the regression error for all other periods. In Example 2 in the previous section, we could infer an obvious violation of that assumption from a visual inspection of a plot of residuals (Exhibit 9). The log-linear trend model of Example 3 appeared to fit the data much better, but we still need to confirm that the uncorrelated errors assumption is satisfied. To address that question formally, we must carry out a Durbin–Watson test on the residuals.

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LOGICAL ORDERING OF TIME-SERIES OBSERVATIONS

In contrast to cross-sectional observations, time-series observations have a logical ordering. They must be processed in chronological order of the time periods involved. For example, we should not make a prediction of the inflation rate using a CPI series in which the order of the observations had been scrambled, because time patterns such as growth in the independent variables can negatively affect the statistical properties of the estimated regression coefficients.

In the reading on regression analysis, we showed how to test whether regression errors are serially correlated using the Durbin–Watson statistic. For example, if the trend models shown in Examples 1 and 3 really capture the time-series behavior of inflation and the log of Starbucks' sales, then the Durbin–Watson statistic for both of those models should not differ significantly from 2.0. Otherwise, the errors in the model are either positively or negatively serially correlated, and that correlation can be used to build a better forecasting model for those time series.

In Example 1, estimating a linear trend in the monthly CPI inflation yielded a Durbin–Watson statistic of 1.2145. Is this result significantly different from 2.0? To find out, we need to test the null hypothesis of no positive serial correlation. For a sample with 228 observations and one independent variable, the critical value, d_l, for the Durbin–Watson test statistic at the 0.05 significance level is above 1.77. Because the value of the Durbin–Watson statistic (1.2145) is below this critical value, we can reject the hypothesis of no positive serial correlation in the errors. (Remember that significantly small values of the Durbin–Watson statistic indicate positive serial correlation; significantly large values point to negative serial correlation; here the Durbin–Watson statistic of 1.09 indicates positive serial correlation.) We can conclude that a regression equation that uses a linear trend to model inflation has positive serial correlation in the errors. We will need a different kind of regression model because this one violates the least squares assumption of no serial correlation in the errors.

In Example 3, estimating a linear trend with the natural logarithm of sales for the Starbucks example yielded a Durbin–Watson statistic of 0.26. Suppose we wish to test the null hypothesis of no positive serial correlation. The critical value, d_l, is above 1.60 at the 0.05 significance level. The value of the Durbin–Watson statistic (0.12) is below this critical value, so we can reject the null hypothesis of no positive serial correlation in the errors. We can conclude that a regression equation that uses a trend to model the log of Starbucks' quarterly sales has positive serial correlation in the errors. So, for this series as well, we need to build a different kind of model.

Overall, we conclude that the trend models sometimes have the limitation that errors are serially correlated. Existence of serial correlation suggests that we can build better forecasting models for such time series than trend models.

5 AR TIME-SERIES MODELS AND COVARIANCE-STATIONARY SERIES

☐ explain the requirement for a time series to be covariance stationary and describe the significance of a series that is not stationary

A key feature of the log-linear model's depiction of time series, and a key feature of time series in general, is that current-period values are related to previous-period values. For example, Starbucks' sales for the current period are related to its sales in

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the previous period. An **autoregressive model (AR)**, a time series regressed on its own past values, represents this relationship effectively. When we use this model, we can drop the normal notation of y as the dependent variable and x as the independent variable because we no longer have that distinction to make. Here we simply use x_t. For example, Equation 4 shows a first-order autoregression, AR(1), for the variable x_t:

$$x_t = b_0 + b_1 x_{t-1} + \varepsilon_t. \tag{4}$$

Thus, in an AR(1) model, we use only the most recent past value of x_t to predict the current value of x_t. In general, a pth-order autoregression, AR(p), for the variable x_t is shown by

$$x_t = b_0 + b_1 x_{t-1} + b_2 x_{t-2} + \ldots + b_p x_{t-p} + \varepsilon_t. \tag{5}$$

In this equation, p past values of x_t are used to predict the current value of x_t. In the next section, we discuss a key assumption of time-series models that include lagged values of the dependent variable as independent variables.

Covariance-Stationary Series

Note that the independent variable (x_{t-1}) in Equation 4 is a random variable. This fact may seem like a mathematical subtlety, but it is not. If we use ordinary least squares to estimate Equation 4 when we have a randomly distributed independent variable that is a lagged value of the dependent variable, our statistical inference may be invalid. To make a valid statistical inference, we must make a key assumption in time-series analysis: We must assume that the time series we are modeling is **covariance stationary**.[1]

What does it mean for a time series to be covariance stationary? The basic idea is that a time series is covariance stationary if its properties, such as mean and variance, do not change over time. A covariance stationary series must satisfy three principal requirements. First, the expected value of the time series must be constant and finite in all periods: $E(y_t) = \mu$ and $|\mu| < \infty$, $t = 1, 2, \ldots, T$ (for this first requirement, we use the absolute value to rule out the case in which the mean is negative without limit—i.e., minus infinity). Second, the variance of the time series must be constant and finite in all periods. Third, the covariance of the time series with itself for a fixed number of periods in the past or future must be constant and finite in all periods. The second and third requirements can be summarized as follows:

$$\text{cov}(y_t, y_{t-s}) = \lambda_s;\ |\lambda_s| < \infty;\ t = 1, 2, \ldots, T;\ s = 0, \pm 1, \pm 2, \ldots, \pm T,$$

where λ signifies a constant. (Note that when s in this equation equals 0, this equation imposes the condition that the variance of the time series is finite, because the covariance of a random variable with itself is its variance: $\text{cov}(y_t, y_t) = \text{var}(y_t)$.) What happens if a time series is not covariance stationary but we model it using Equation 4? The estimation results will have no economic meaning. For a non-covariance-stationary time series, estimating the regression in Equation 4 will yield spurious results. In particular, the estimate of b_1 will be biased, and any hypothesis tests will be invalid.

How can we tell if a time series is covariance stationary? We can often answer this question by looking at a plot of the time series. If the plot shows roughly the same mean and variance over time without any significant seasonality, then we may want to assume that the time series is covariance stationary.

Some of the time series we looked at in the exhibits appear to be covariance stationary. For example, the inflation data shown in Exhibit 3 appear to have roughly the same mean and variance over the sample period. Many of the time series one

1 "Weakly stationary" is a synonym for covariance stationary. Note that the terms "stationary" and "stationarity" are often used to mean "covariance stationary" or "covariance stationarity," respectively. You may also encounter the more restrictive concept of "strictly" stationary, which has little practical application. For details, see Diebold (2008).

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encounters in business and investments, however, are not covariance stationary. For example, many time series appear to grow (or decline) steadily over time and thus have a mean that is nonconstant, which implies that they are nonstationary. As an example, the time series of quarterly sales in Exhibit 8 clearly shows the mean increasing as time passes. Thus, Starbucks' quarterly sales are not covariance stationary (in general, any time series accurately described with a linear or log-linear trend model is not covariance stationary, although a transformation of the original series might be covariance stationary). Macroeconomic time series such as those relating to income and consumption are often strongly trending as well. A time series with seasonality (regular patterns of movement with the year) also has a nonconstant mean, as do other types of time series that we discuss later (in particular, random walks are not covariance stationary).

Exhibit 2 showed that monthly retail sales (not seasonally adjusted) are also not covariance stationary. Sales in December are always much higher than sales in other months (these are the regular large peaks), and sales in January are always much lower (these are the regular large drops after the December peaks). On average, sales also increase over time, so the mean of sales is not constant.

Later we will show that we can often transform a nonstationary time series into a stationary time series. But whether a stationary time series is original or transformed, a warning is necessary: Stationarity in the past does not guarantee stationarity in the future. There is always the possibility that a well-specified model will fail when the state of the world changes and yields a different underlying model that generates the time series.

6 DETECTING SERIALLY CORRELATED ERRORS IN AN AR MODEL

- ☐ describe the structure of an autoregressive (AR) model of order p and calculate one- and two-period-ahead forecasts given the estimated coefficients
- ☐ explain how autocorrelations of the residuals can be used to test whether the autoregressive model fits the time series

We can estimate an autoregressive model using ordinary least squares if the time series is covariance stationary and the errors are uncorrelated. Unfortunately, our previous test for serial correlation, the Durbin–Watson statistic, is invalid when the independent variables include past values of the dependent variable. Therefore, for most time-series models, we cannot use the Durbin–Watson statistic. Fortunately, we can use other tests to determine whether the errors in a time-series model are serially correlated. One such test reveals whether the autocorrelations of the error term are significantly different from 0. This test is a *t*-test involving a residual autocorrelation and the standard error of the residual autocorrelation. As background for the test, we next discuss autocorrelation in general before moving to residual autocorrelation.

The **autocorrelations** of a time series are the correlations of that series with its own past values. The order of the correlation is given by k, where k represents the number of periods lagged. When $k = 1$, the autocorrelation shows the correlation of the variable in one period with its occurrence in the previous period. For example, the ***k*th-order autocorrelation** (ρ_k) is

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$$\rho_k = \frac{\text{cov}(x_t, x_{t-k})}{\sigma_x^2} = \frac{E\left[(x_t - \mu)\ (x_{t-k} - \mu)\right]}{\sigma_x^2},$$

where E stands for the expected value. Note that we have the relationship $\text{cov}(x_t, x_{t-k}) \leq \sigma_x^2$, with equality holding when $k = 0$. This means that the absolute value of ρ_k is less than or equal to 1.

Of course, we can never directly observe the autocorrelations, ρ_k. Instead, we must estimate them. Thus, we replace the expected value of x_t, μ, with its estimated value, $\bar{x}$, to compute the estimated autocorrelations. The kth-order estimated autocorrelation of the time series x_t, which we denote $\hat{\rho}_k$, is

$$\hat{\rho}_k = \frac{\sum_{t=k+1}^{T}\left[(x_t - \bar{x})\ (x_{t-k} - \bar{x})\right]}{\sum_{t=1}^{T}(x_t - \bar{x})^2}.$$

Analogous to the definition of autocorrelations for a time series, we can define the autocorrelations of the error term for a time-series model as[2]

$$\begin{aligned}\rho_{\varepsilon,k} &= \frac{\text{cov}(\varepsilon_t, \varepsilon_{t-k})}{\sigma_\varepsilon^2} \\ &= \frac{E\left[(\varepsilon_t - 0)\ (\varepsilon_{t-k} - 0)\right]}{\sigma_\varepsilon^2} \\ &= \frac{E(\varepsilon_t \varepsilon_{t-k})}{\sigma_\varepsilon^2}.\end{aligned}$$

We assume that the expected value of the error term in a time-series model is 0.[3]

We can determine whether we are using the correct time-series model by testing whether the autocorrelations of the error term (**error autocorrelations**) differ significantly from 0. If they do, the model is not specified correctly. We estimate the error autocorrelation using the sample autocorrelations of the residuals (**residual autocorrelations**) and their sample variance.

A test of the null hypothesis that an error autocorrelation at a specified lag equals 0 is based on the residual autocorrelation for that lag and the standard error of the residual correlation, which is equal to $1/\sqrt{T}$, where T is the number of observations in the time series (Diebold 2008). Thus, if we have 100 observations in a time series, the standard error for each of the estimated autocorrelations is 0.1. We can compute the t-test of the null hypothesis that the error correlation at a particular lag equals 0 by dividing the residual autocorrelation at that lag by its standard error $\left(1/\sqrt{T}\right)$.

How can we use information about the error autocorrelations to determine whether an autoregressive time-series model is correctly specified? We can use a simple three-step method. First, estimate a particular autoregressive model—say, an AR(1) model. Second, compute the autocorrelations of the residuals from the model.[4] Third, test to see whether the residual autocorrelations differ significantly from 0. If significance tests show that the residual autocorrelations differ significantly from 0,

2 Whenever we refer to autocorrelation without qualification, we mean autocorrelation of the time series itself rather than autocorrelation of the error term or residuals.

3 This assumption is similar to the one made in earlier coverage of regression analysis about the expected value of the error term.

4 We can compute these residual autocorrelations easily with most statistical software packages. In Microsoft Excel, for example, to compute the first-order residual autocorrelation, we compute the correlation of the residuals from Observations 1 through $T - 1$ with the residuals from Observations 2 through T.

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the model is not correctly specified; we may need to modify it in ways that we will discuss shortly.[5] We now present an example to demonstrate how this three-step method works.

EXAMPLE 4

Predicting Gross Margins for Intel Corporation

1. Analyst Melissa Jones decides to use a time-series model to predict Intel Corporation's gross margin [(Sales – Cost of goods sold)/Sales] using quarterly data from the first quarter of 2003 through the second quarter of 2019. She does not know the best model for gross margin but believes that the current-period value will be related to the previous-period value. She decides to start out with a first-order autoregressive model, AR(1): Gross margin$_t$ = b_0 + b_1(Gross margin$_{t-1}$) + ε_t. Her observations on the dependent variable are 1Q 2003 through 2Q 2019. Exhibit 12 shows the results of estimating this AR(1) model, along with the autocorrelations of the residuals from that model.

Exhibit 12: Autoregression: AR(1) Model Gross Margin of Intel Quarterly Observations, January 2003–June 2019

Regression Statistics	
R^2	0.5746
Standard error	0.03002
Observations	65
Durbin–Watson	1.743

	Coefficient	Standard Error	*t*-Statistic
Intercept	0.1513	0.0480	3.15
Gross margin$_{t-1}$	0.7462	0.0809	9.2236

Autocorrelations of the Residual			
Lag	**Autocorrelation**	**Standard Error**	***t*-Statistic**
1	0.1308	0.1240	1.0545
2	–0.2086	0.1240	–1.6818
3	0.0382	0.1240	0.3080
4	0.0608	0.1240	0.4903

Source: Bloomberg.

The first thing to note about Exhibit 12 is that both the intercept ($\hat{b}_0$ = 0.1513) and the coefficient on the first lag ($\hat{b}_1$ = 0.7462) of the gross margin are highly significant in the regression equation. The first lag of a time series is the value of the time series in the previous period. The *t*-statistic for the

5 Often, econometricians use additional tests for the significance of residual autocorrelations. For example, the Box–Pierce *Q*-statistic is frequently used to test the joint hypothesis that all autocorrelations of the residuals are equal to 0. For further discussion, see Diebold (2008).

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intercept is about 3.2, whereas the *t*-statistic for the first lag of the gross margin is more than 9. With 65 observations and two parameters, this model has 63 degrees of freedom. At the 0.05 significance level, the critical value for a *t*-statistic is about 2.0. Therefore, Jones must reject the null hypotheses that the intercept is equal to 0 ($b_0 = 0$) and the coefficient on the first lag is equal to 0 ($b_1 = 0$) in favor of the alternative hypothesis that the coefficients, individually, are not equal to 0. But are these statistics valid? Although the Durbin–Watson statistic is presented in Exhibit 12, it cannot be used to test serial correlation when the independent variables include past values of the dependent variable. The correct approach is to test whether the residuals from this model are serially correlated.

At the bottom of Exhibit 12, the first four autocorrelations of the residual are displayed along with the standard error and the *t*-statistic for each of those autocorrelations.[6] The sample has 65 observations, so the standard error for each of the autocorrelations is $1/\sqrt{65} = 0.1240$. Exhibit 12 shows that none of the first four autocorrelations has a *t*-statistic larger than 1.6818 in absolute value. Therefore, Jones can conclude that none of these autocorrelations differs significantly from 0. Consequently, she can assume that the residuals are not serially correlated and that the model is correctly specified, and she can validly use ordinary least squares to estimate the parameters and the parameters' standard errors in the autoregressive model (for other tests for serial correlation of residuals, see Diebold 2008).

Now that Jones has concluded that this model is correctly specified, how can she use it to predict Intel's gross margin in the next period? The estimated equation is $\text{Gross margin}_t = 0.1513 + 0.7462(\text{Gross margin}_{t-1}) + \varepsilon_t$. The expected value of the error term is 0 in any period. Therefore, this model predicts that gross margin in period $t + 1$ will be $\text{Gross margin}_{t+1} = 0.1513 + 0.7462(\text{Gross margin}_t)$. For example, if gross margin is 55% in this quarter (0.55), the model predicts that in the next quarter gross margin will increase to 0.1513 + 0.7462(0.55) = 0.5617, or 56.17%. However, if gross margin is currently 65% (0.65), the model predicts that in the next quarter, gross margin will fall to 0.1513 + 0.7462(0.65) = 0.6363, or 63.63%. As we show in the following section, the model predicts that gross margin will increase if it is below a certain level (59.61%) and decrease if it is above that level.

MEAN REVERSION AND MULTIPERIOD FORECASTS

7

- ☐ explain mean reversion and calculate a mean-reverting level
- ☐ describe the structure of an autoregressive (AR) model of order p and calculate one- and two-period-ahead forecasts given the estimated coefficients

6 For seasonally unadjusted data, analysts often compute the same number of autocorrelations as there are observations in a year (for example, four for quarterly data). The number of autocorrelations computed also often depends on sample size, as discussed in Diebold (2008).

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We say that a time series shows **mean reversion** if it tends to fall when its level is above its mean and rise when its level is below its mean. Much like the temperature in a room controlled by a thermostat, a mean-reverting time series tends to return to its long-term mean. How can we determine the value that the time series tends toward? If a time series is currently at its mean-reverting level, then the model predicts that the value of the time series will be the same in the next period. At its mean-reverting level, we have the relationship $x_{t+1} = x_t$. For an AR(1) model ($x_{t+1} = b_0 + b_1 x_t$), the equality $x_{t+1} = x_t$ implies the level $x_t = b_0 + b_1 x_t$ or that the mean-reverting level, x_t, is given by

$$x_t = \frac{b_0}{1 - b_1}.$$

So the AR(1) model predicts that the time series will stay the same if its current value is $b_0/(1 - b_1)$, increase if its current value is below $b_0/(1 - b_1)$, and decrease if its current value is above $b_0/(1 - b_1)$.

In the case of gross margins for Intel, the mean-reverting level for the model shown in Exhibit 12 is 0.1513/(1 – 0.7462) = 0.5961. If the current gross margin is above 0.5961, the model predicts that the gross margin will fall in the next period. If the current gross margin is below 0.5961, the model predicts that the gross margin will rise in the next period. As we will discuss later, all covariance-stationary time series have a finite mean-reverting level.

Multiperiod Forecasts and the Chain Rule of Forecasting

Often, financial analysts want to make forecasts for more than one period. For example, we might want to use a quarterly sales model to predict sales for a company for each of the next four quarters. To use a time-series model to make forecasts for more than one period, we must examine how to make multiperiod forecasts using an AR(1) model. The one-period-ahead forecast of x_t from an AR(1) model is as follows:

$$\hat{x}_{t+1} = \hat{b}_0 + \hat{b}_1 x_t \tag{6}$$

If we want to forecast x_{t+2} using an AR(1) model, our forecast will be based on

$$\hat{x}_{t+2} = \hat{b}_0 + \hat{b}_1 x_{t+1} \tag{7}$$

Unfortunately, we do not know x_{t+1} in period *t,* so we cannot use Equation 7 directly to make a two-period-ahead forecast. We can, however, use our forecast of x_{t+1} and the AR(1) model to make a prediction of x_{t+2}. The **chain rule of forecasting** is a process in which the next period's value, predicted by the forecasting equation, is substituted into the equation to give a predicted value two periods ahead. Using the chain rule of forecasting, we can substitute the predicted value of x_{t+1} into Equation 7 to get $\hat{x}_{t+2} = \hat{b}_0 + \hat{b}_1 \hat{x}_{t+1}$. We already know $\hat{x}_{t+1}$ from our one-period-ahead forecast in Equation 6. Now we have a simple way of predicting x_{t+2}.

Multiperiod forecasts are more uncertain than single-period forecasts because each forecast period has uncertainty. For example, in forecasting x_{t+2}, we first have the uncertainty associated with forecasting x_{t+1} using x_t, and then we have the uncertainty associated with forecasting x_{t+2} using the forecast of x_{t+1}. In general, the more periods a forecast has, the more uncertain it is. Note that if a forecasting model is well specified, the prediction errors from the model will not be serially correlated. If the prediction errors for each period are not serially correlated, then the variance of a multiperiod forecast will be higher than the variance of a single-period forecast.

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EXAMPLE 5

Multiperiod Prediction of Intel's Gross Margin

Suppose that at the beginning of 2020, we want to predict Intel's gross margin in two periods using the model shown in Exhibit 12. Assume that Intel's gross margin in the current period is 63%. The one-period-ahead forecast of Intel's gross margin from this model is 0.6214 = 0.1513 + 0.7462(0.63). By substituting the one-period-ahead forecast, 0.6214, back into the regression equation, we can derive the following two-period-ahead forecast: 0.6150 = 0.1513 + 0.7462(0.6214). Therefore, if the current gross margin for Intel is 63%, the model predicts that Intel's gross margin in two quarters will be 61.50%.

EXAMPLE 6

Modeling US CPI Inflation

Analyst Lisette Miller has been directed to build a time-series model for monthly US inflation. Inflation and expectations about inflation, of course, have a significant effect on bond returns. For a 24-year period beginning January 1995 and ending December 2018, she selects as data the annualized monthly percentage change in the CPI. Which model should Miller use?

The process of model selection parallels that of Example 4 relating to Intel's gross margins. The first model Miller estimates is an AR(1) model, using the previous month's inflation rate as the independent variable: $\text{Inflation}_t = b_0 + b_1(\text{Inflation}_{t-1}) + \varepsilon_t$, $t = 1, 2, \ldots, 287$. To estimate this model, she uses monthly CPI inflation data from January 1995 to December 2018 ($t = 1$ denotes February 1995). Exhibit 13 shows the results of estimating this model.

Exhibit 13: Monthly CPI Inflation at an Annual Rate: AR(1) Model—Monthly Observations, February 1995–December 2018

Regression Statistics	
R^2	0.1586
Standard error	2.9687
Observations	287
Durbin–Watson	1.8442

	Coefficient	Standard Error	*t*-Statistic
Intercept	0.13346	0.2134	0.6254
Inflation_{t-1}	0.3984	0.0544	7.3235

Autocorrelations of the Residual			
Lag	**Autocorrelation**	**Standard Error**	***t*-Statistic**
1	0.0777	0.0590	1.3175
2	−0.1653	0.0590	−2.8013
3	−0.1024	0.0590	−1.7362
4	−0.0845	0.0590	1.4324

Source: US Bureau of Labor Statistics.

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As Exhibit 13 shows, both the intercept ($\hat{b}_0 = 1.3346$) and the coefficient on the first lagged value of inflation ($\hat{b}_1 = 0.3984$) are highly statistically significant, with large *t*-statistics. With 287 observations and two parameters, this model has 285 degrees of freedom. The critical value for a *t*-statistic at the 0.05 significance level is about 1.97. Therefore, Miller can reject the individual null hypotheses that the intercept is equal to 0 ($b_0 = 0$) and the coefficient on the first lag is equal to 0 ($b_1 = 0$) in favor of the alternative hypothesis that the coefficients, individually, are not equal to 0.

Are these statistics valid? Miller will know when she tests whether the residuals from this model are serially correlated. With 287 observations in this sample, the standard error for each of the estimated autocorrelations is $1/\sqrt{287} = 0.0590$. The critical value for the *t*-statistic is 1.97. Because the second estimated autocorrelation has *t*-statistic larger than 1.97 in absolute value, Miller concludes that the autocorrelations are significantly different from 0. This model is thus misspecified because the residuals are serially correlated.

If the residuals in an autoregressive model are serially correlated, Miller can eliminate the correlation by estimating an autoregressive model with more lags of the dependent variable as explanatory variables. Exhibit 14 shows the result of estimating a second time-series model, an AR(2) model using the same data as in the analysis shown in Exhibit 13. With 286 observations and three parameters, this model has 283 degrees of freedom. Because the degrees of freedom are almost the same as those for the estimates shown in Exhibit 13, the critical value of the *t*-statistic at the 0.05 significance level also is almost the same (1.97). If she estimates the equation with two lags—$\text{Inflation}_t = b_0 + b_1(\text{Inflation}_{t-1}) + b_2(\text{Inflation}_{t-2}) + \varepsilon_t$—Miller finds that all three of the coefficients in the regression model (an intercept and the coefficients on two lags of the dependent variable) differ significantly from 0. The bottom portion of Exhibit 14 shows that none of the first four autocorrelations of the residual has a *t*-statistic greater in absolute value than the critical value of 1.97. Therefore, Miller fails to reject the hypothesis that the individual autocorrelations of the residual equal 0. She concludes that this model is correctly specified because she finds no evidence of serial correlation in the residuals.

Exhibit 14: Monthly CPI Inflation at an Annual Rate: AR(2) Model—Monthly Observations, March 1995–December 2018

Regression Statistics	
R^2	0.1907
Standard error	2.9208

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Regression Statistics	
Observations	286
Durbin–Watson	1.9934

	Coefficient	Standard Error	*t*-Statistic
Intercept	1.5996	0.2245	7.1252
Inflation_{t-1}	0.4759	0.0583	8.1636
Inflation_{t-2}	–0.1964	0.0583	–3.368

Autocorrelations of the Residual

Lag	Autocorrelation	Standard Error	*t*-Statistic
1	0.0032	0.0591	0.0536
2	0.0042	0.0591	0.0707
3	–0.0338	0.0591	–0.5696
4	0.0155	0.0591	0.2623

Source: US Bureau of Labor Statistics.

1. The analyst selected an AR(2) model because the residuals from the AR(1) model were serially correlated. Suppose that in a given month, inflation had been 4% at an annual rate in the previous month and 3% in the month before that. What would be the difference in the analyst forecast of inflation for that month if she had used an AR(1) model instead of the AR(2) model?

Solution:

The AR(1) model shown in Exhibit 13 predicted that inflation in the next month would be 1.3346 + 0.3984(4) = 2.93%, approximately, whereas the AR(2) model shown in Exhibit 14 predicts that inflation in the next month will be 1.5996 + 0.4759(4) – 0.1964(3) = 2.91% approximately. If the analyst had used the incorrect AR(1) model, she would have predicted inflation to be 2 bps higher (2.93% versus 2.91%) than when using the AR(2) model. Although in this case the difference in the predicted inflation is actually very small, this kind of scenario illustrates that using an incorrect forecast could adversely affect the quality of her company's investment choices.

COMPARING FORECAST MODEL PERFORMANCE

8

☐ contrast in-sample and out-of-sample forecasts and compare the forecasting accuracy of different time-series models based on the root mean squared error criterion

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One way to compare the forecast performance of two models is to compare the variance of the forecast errors that the two models make. The model with the smaller forecast error variance will be the more accurate model, and it will also have the smaller standard error of the time-series regression. (This standard error usually is reported directly in the output for the time-series regression.)

In comparing forecast accuracy among models, we must distinguish between in-sample forecast errors and out-of-sample forecast errors. **In-sample forecast errors** are the residuals from a fitted time-series model. For example, when we estimated a linear trend with raw inflation data from January 1995 to December 2018, the in-sample forecast errors were the residuals from January 1995 to December 2018. If we use this model to predict inflation outside this period, the differences between actual and predicted inflation are **out-of-sample forecast errors**.

EXAMPLE 7

In-Sample Forecast Comparisons of US CPI Inflation

In Example 6, the analyst compared an AR(1) forecasting model of monthly US inflation with an AR(2) model of monthly US inflation and decided that the AR(2) model was preferable. Exhibit 13 showed that the standard error from the AR(1) model of inflation is 2.9687, and Exhibit 14 showed that the standard error from the AR(2) model is 2.9208. Therefore, the AR(2) model had a lower in-sample forecast error variance than the AR(1) model had, which is consistent with our belief that the AR(2) model was preferable. Its standard error is 2.9208/2.9687 = 98.39% of the forecast error of the AR(1) model.

Often, we want to compare the forecasting accuracy of different models after the sample period for which they were estimated. We wish to compare the out-of-sample forecast accuracy of the models. Out-of-sample forecast accuracy is important because the future is always out of sample. Although professional forecasters distinguish between out-of-sample and in-sample forecasting performance, many articles that analysts read contain only in-sample forecast evaluations. Analysts should be aware that out-of-sample performance is critical for evaluating a forecasting model's real-world contribution.

Typically, we compare the out-of-sample forecasting performance of forecasting models by comparing their **root mean squared error (RMSE)**, which is the square root of the average squared error. The model with the smallest RMSE is judged the most accurate. The following example illustrates the computation and use of RMSE in comparing forecasting models.

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EXAMPLE 8

Out-of-Sample Forecast Comparisons of US CPI Inflation

1. Suppose we want to compare the forecasting accuracy of the AR(1) and AR(2) models of US inflation estimated over 1995 to 2018, using CPI data from January 2019 to September 2019.

Exhibit 15: Out-of-Sample Forecast Error Comparisons: January 2019–September 2019 US CPI Inflation (Annualized)

Date	Infl(t)	Infl($t-1$)	Infl($t-2$)	AR(1) Error	Squared Error	AR(2) Error	Squared Error
2019							
January	0.0000	0.0000	0.0000	0.1335	0.0178	−1.6000	2.5599
February	2.4266	0.0000	0.0000	−2.2931	5.2585	0.8266	0.6833
March	4.9070	2.4266	0.0000	−3.8068	14.4916	2.1522	4.6320
April	3.6600	4.9070	2.4266	−1.5716	2.4699	0.2014	0.0406
May	1.2066	3.6600	4.9070	0.3850	0.1482	−1.1714	1.3722
June	1.2066	1.2066	3.6600	−0.5924	0.3510	−0.2488	0.0619
July	3.6600	1.2066	1.2066	−3.0458	9.2770	1.7228	2.9680
August	1.2066	3.6600	1.2066	0.3850	0.1482	−1.8982	3.6030
September	0.0000	1.2066	3.6600	0.6142	0.3772	−1.4554	2.1181
				Average	3.6155	Average	2.0043
				RMSE	1.9014	RMSE	1.4157

Note: Any apparent discrepancies between error and squared error results are due to rounding.

Source: US Bureau of Labor Statistics.

Solution:

For each month from January 2019 to September 2019, the first column of numbers in Exhibit 15 shows the actual annualized inflation rate during the month. The second and third columns show the rate of inflation in the previous two months. The fourth column shows the out-of-sample errors (Actual – Forecast) from the AR(1) model shown in Exhibit 13. The fifth column shows the squared errors from the AR(1) model. The sixth column shows the out-of-sample errors from the AR(2) model shown in Exhibit 14. The final column shows the squared errors from the AR(2) model. The bottom of the table displays the average squared error and the RMSE. According to these measures, the AR(2) model was slightly more accurate than the AR(1) model in its out-of-sample forecasts of inflation from January 2019 to September 2019. The RMSE from the AR(2) model was only 1.4157/1.9014 = 74.46% as large as the RMSE from the AR(1) model. Therefore, the AR(2) model was more accurate both in sample and out of sample. Of course, this was a small sample to use in evaluating out-of-sample forecasting performance. Sometimes, an analyst may have conflicting information about whether to choose an AR(1) or an AR(2) model. We must also consider regression coefficient stability. We will continue the comparison between these two models in the following section.

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9 INSTABILITY OF REGRESSION COEFFICIENTS

☐ explain the instability of coefficients of time-series models

One of the important issues an analyst faces in modeling a time series is the sample period to use. The estimates of regression coefficients of the time-series model can change substantially across different sample periods used for estimating the model. Often, the regression coefficient estimates of a time-series model estimated using an earlier sample period can be quite different from those of a model estimated using a later sample period. Similarly, the estimates can be different between models estimated using relatively shorter and longer sample periods. Further, the choice of model for a particular time series can also depend on the sample period. For example, an AR(1) model may be appropriate for the sales of a company in one particular sample period, but an AR(2) model may be necessary for an earlier or later sample period (or for a longer or shorter sample period). Thus, the choice of a sample period is an important decision in modeling a financial time series.

Unfortunately, there is usually no clear-cut basis in economic or financial theory for determining whether to use data from a longer or shorter sample period to estimate a time-series model. We can get some guidance, however, if we remember that our models are valid only for covariance-stationary time series. For example, we should not combine data from a period when exchange rates were fixed with data from a period when exchange rates were floating. The exchange rates in these two periods would not likely have the same variance because exchange rates are usually much more volatile under a floating-rate regime than when rates are fixed. Similarly, many US analysts consider it inappropriate to model US inflation or interest-rate behavior since the 1960s as a part of one sample period, because the Federal Reserve had distinct policy regimes during this period. A simple way to determine appropriate samples for time-series estimation is to look at graphs of the data to see whether the time series looks stationary before estimation begins. If we know that a government policy changed on a specific date, we might also test whether the time-series relation was the same before and after that date.

In the following example, we illustrate how the choice of a longer versus a shorter period can affect the decision of whether to use, for example, a first- or second-order time-series model. We then show how the choice of the time-series model (and the associated regression coefficients) affects our forecast. Finally, we discuss which sample period, and accordingly which model and corresponding forecast, is appropriate for the time series analyzed in the example.

EXAMPLE 9

Instability in Time-Series Models of US Inflation

In Example 6, the analyst Lisette Miller concluded that US CPI inflation should be modeled as an AR(2) time series. A colleague examined her results and questioned estimating one time-series model for inflation in the United States since 1995, given that the Federal Reserve responded aggressively to the financial crisis that emerged in 2007. He argues that the inflation time series from 1995 to 2018 has two **regimes** or underlying models generating the time series: one running from 1995 through 2007 and another starting in 2008. Therefore, the colleague suggests that Miller estimate a new time-series model for US inflation

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starting in 2008. Because of his suggestion, Miller first estimates an AR(1) model for inflation using data for a sample period from 2008 to 2018. Exhibit 16 shows her AR(1) estimates.

Exhibit 16: Autoregression: AR(1) Model Monthly CPI Inflation at an Annual Rate, January 2008–December 2018

Regression Statistics	
R^2	0.2536
Standard error	3.0742
Observations	132
Durbin–Watson	1.8164

	Coefficient	Standard Error	*t*-Statistic
Intercept	0.8431	0.2969	2.8397
Inflation_{t-1}	0.5036	0.0758	6.6438

Autocorrelations of the Residual			
Lag	**Autocorrelation**	**Standard Error**	***t*-Statistic**
1	0.0999	0.087	1.1479
2	−0.1045	0.087	−1.2015
3	−0.1568	0.087	−1.8051
4	0.0500	0.087	0.5750

Source: US Bureau of Labor Statistics.

The bottom part of Exhibit 16 shows that the first four autocorrelations of the residuals from the AR(1) model are quite small. None of these autocorrelations has a *t*-statistic larger than 1.99, the critical value for significance. Consequently, Miller cannot reject the null hypothesis that the residuals are serially uncorrelated. The AR(1) model is correctly specified for the sample period from 2008 to 2018, so there is no need to estimate the AR(2) model. This conclusion is very different from that reached in Example 6 using data from 1995 to 2018. In that example, Miller initially rejected the AR(1) model because its residuals exhibited serial correlation. When she used a larger sample, an AR(2) model initially appeared to fit the data much better than did an AR(1) model.

How deeply does our choice of sample period affect our forecast of future inflation? Suppose that in a given month, inflation was 4% at an annual rate, and the month before that it was 3%. The AR(1) model shown in Exhibit 16 predicts that inflation in the next month will be 0.8431 + 0.5036(4) ≈ 2.86%. Therefore, the forecast of the next month's inflation using the 2008 to 2018 sample is 2.86%. Remember from the analysis following Example 6 that the AR(2) model for the 1995 to 2018 sample predicts inflation of 2.91% in the next month. Thus, using the correctly specified model for the shorter sample produces an inflation forecast 0.05 pps below the forecast made from the correctly specified model for the longer sample period. Such a difference might substantially affect a particular investment decision.

Which model is correct? Exhibit 17 suggests an answer. Monthly US inflation was so much more volatile during the middle part of the study period than in the earlier or later years that inflation is probably not a covariance-stationary time series from 1995 to 2018. Therefore, we can reasonably believe that the

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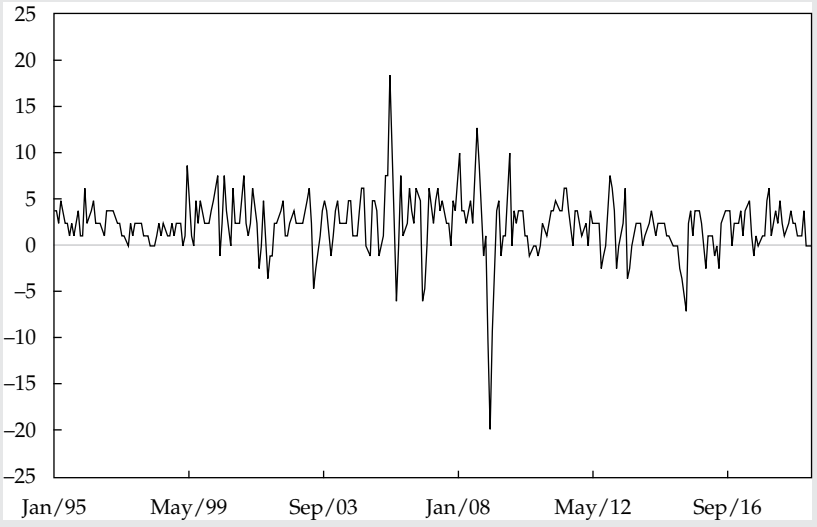
data have more than one regime and Miller should estimate a separate model for inflation from 2009 to 2018, as shown previously. In fact, the standard deviation of annualized monthly inflation rates is just 2.86% for 1995–2007 but 3.54% for 2008–2018, largely because of volatility during the 2008 crisis. As the example shows, experience (such as knowledge of government policy changes) and judgment play a vital role in determining how to model a time series. Simply relying on autocorrelations of the residuals from a time-series model cannot tell us the correct sample period for our analysis.

Exhibit 17: Monthly CPI Inflation



Source: US Bureau of Labor Statistics.

10 RANDOM WALKS

- ☐ describe characteristics of random walk processes and contrast them to covariance stationary processes
- ☐ explain mean reversion and calculate a mean-reverting level

So far, we have examined those time series in which the time series has a tendency to revert to its mean level as the change in a variable from one period to the next follows a mean-reverting pattern. In contrast, there are many financial time series in which the changes follow a random pattern. We discuss these "random walks" in the following section.

Random Walks

A random walk is one of the most widely studied time-series models for financial data. A **random walk** is a time series in which the value of the series in one period is the value of the series in the previous period plus an unpredictable random error. A random walk can be described by the following equation:

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$$x_t = x_{t-1} + \varepsilon_t, \quad E(\varepsilon_t) = 0, \quad E(\varepsilon_t^2) = \sigma^2, \quad \text{cov}(\varepsilon_t, \varepsilon_s) = E(\varepsilon_t \varepsilon_s) = 0 \text{ if } t \neq s. \tag{8}$$

Equation 8 means that the time series x_t is in every period equal to its value in the previous period plus an error term, ε_t, that has constant variance and is uncorrelated with the error term in previous periods. Note two important points. First, this equation is a special case of an AR(1) model with $b_0 = 0$ and $b_1 = 1$.[7] Second, the expected value of ε_t is zero. Therefore, the best forecast of x_t that can be made in period $t - 1$ is x_{t-1}. In fact, in this model, x_{t-1} is the best forecast of x in every period after $t - 1$.

Random walks are quite common in financial time series. For example, many studies have tested whether and found that currency exchange rates follow a random walk. Consistent with the second point made in the previous paragraph, some studies have found that sophisticated exchange rate forecasting models cannot outperform forecasts made using the random walk model and that the best forecast of the future exchange rate is the current exchange rate.

Unfortunately, we cannot use the regression methods we have discussed so far to estimate an AR(1) model on a time series that is actually a random walk. To see why this is so, we must determine why a random walk has no finite mean-reverting level or finite variance. Recall that if x_t is at its mean-reverting level, then $x_t = b_0 + b_1 x_t$, or $x_t = b_0/(1 - b_1)$. In a random walk, however, $b_0 = 0$ and $b_1 = 1$, so $b_0/(1 - b_1) = 0/0$. Therefore, a random walk has an undefined mean-reverting level.

What is the variance of a random walk? Suppose that in Period 1, the value of x_1 is 0. Then we know that $x_2 = 0 + \varepsilon_2$. Therefore, the variance of $x_2 = \text{var}(\varepsilon_2) = \sigma^2$. Now $x_3 = x_2 + \varepsilon_3 = \varepsilon_2 + \varepsilon_3$. Because the error term in each period is assumed to be uncorrelated with the error terms in all other periods, the variance of $x_3 = \text{var}(\varepsilon_2) + \text{var}(\varepsilon_3) = 2\sigma^2$. By a similar argument, we can show that for any period t, the variance of $x_t = (t - 1)\sigma^2$. But this means that as t grows large, the variance of x_t grows without an upper bound: It approaches infinity. This lack of upper bound, in turn, means that a random walk is not a covariance-stationary time series, because a covariance-stationary time series must have a finite variance.

What is the practical implication of these issues? *We cannot use standard regression analysis on a time series that is a random walk.* We can, however, attempt to convert the data to a covariance-stationary time series if we suspect that the time series is a random walk. In statistical terms, we can difference it.

We difference a time series by creating a new time series—say, y_t—that in each period is equal to the difference between x_t and x_{t-1}. This transformation is called **first-differencing** because it subtracts the value of the time series in the first prior period from the current value of the time series. Sometimes the first difference of x_t is written as $\Delta x_t = x_t - x_{t-1}$. Note that the first difference of the random walk in Equation 8 yields

$$y_t = x_t - x_{t-1} = \varepsilon_t, \quad E(\varepsilon_t) = 0, \quad E(\varepsilon_t^2) = \sigma^2, \quad \text{cov}(\varepsilon_t, \varepsilon_s) = E(\varepsilon_t \varepsilon_s) = 0 \text{ for } t \neq s.$$

The expected value of ε_t is 0. Therefore, the best forecast of y_t that can be made in period $t - 1$ is 0. This implies that the best forecast is that there will be no change in the value of the current time series, x_{t-1}.

The first-differenced variable, y_t, is covariance stationary. How is this so? First, note that this model ($y_t = \varepsilon_t$) is an AR(1) model with $b_0 = 0$ and $b_1 = 0$. We can compute the mean-reverting level of the first-differenced model as $b_0/(1 - b_1) = 0/1 = 0$. Therefore, a first-differenced random walk has a mean-reverting level of 0. Note also that the variance of y_t in each period is $\text{var}(\varepsilon_t) = \sigma^2$. Because the variance and the mean of y_t

7 Equation 8 with a nonzero intercept added (as in Equation 9, given later) is sometimes referred to as a random walk with drift.

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are constant and finite in each period, y_t is a covariance-stationary time series and we can model it using linear regression. Of course, modeling the first-differenced series with an AR(1) model does not help us predict the future, because $b_0 = 0$ and $b_1 = 0$. We simply conclude that the original time series is, in fact, a random walk.

Had we tried to estimate an AR(1) model for a time series that was a random walk, our statistical conclusions would have been incorrect because AR models cannot be used to estimate random walks or any time series that is not covariance stationary. The following example illustrates this issue with exchange rates.

EXAMPLE 10

The Yen/US Dollar Exchange Rate

1. Financial analysts often assume that exchange rates are random walks. Consider an AR(1) model for the Japanese yen/US dollar exchange rate (JPY/USD). Exhibit 18 shows the results of estimating the model using month-end observations from October 1980 through August 2019.

Exhibit 18: Yen/US Dollar Exchange Rate: AR(1) Model Month-End Observations, October 1980–August 2019

Regression Statistics	
R^2	0.9897
Standard error	4.5999
Observations	467
Durbin–Watson	1.9391

	Coefficient	Standard Error	*t*-Statistic
Intercept	0.8409	0.6503	1.2931
JPY/USD_{t-1}	0.9919	0.0047	211.0426

Autocorrelations of the Residual			
Lag	**Autocorrelation**	**Standard Error**	***t*-Statistic**
1	0.0302	0.0465	0.6495
2	0.0741	0.0465	1.5935
3	0.0427	0.0465	0.9183
4	–0.0034	0.0465	0.0731

Source: US Federal Reserve Board of Governors.

The results in Exhibit 18 suggest that the yen/US dollar exchange rate is a random walk because the estimated intercept does not appear to be significantly different from 0 and the estimated coefficient on the first lag of the exchange rate is very close to 1. Can we use the *t*-statistics in Exhibit 18 to test whether the exchange rate is a random walk? Unfortunately, no, because the standard errors in an AR model are invalid if the model is estimated using a data series that is a random walk (remember, a random walk is not covariance stationary). If the exchange rate is, in fact, a random walk, we might come to an incorrect conclusion based on faulty statistical tests and

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then invest incorrectly. We can use a test presented in the next section to test whether the time series is a random walk.

Suppose the exchange rate is a random walk, as we now suspect. If so, the first-differenced series, $y_t = x_t - x_{t-1}$, will be covariance stationary. We present the results from estimating $y_t = b_0 + b_1 y_{t-1} + \varepsilon_t$ in Exhibit 19. If the exchange rate is a random walk, then $b_0 = 0$, $b_1 = 0$, and the error term will not be serially correlated.

Exhibit 19: First-Differenced Yen/US Dollar Exchange Rate: AR(1) Model Month-End Observations, November 1980–August 2019

Regression Statistics	
R^2	0.0008
Standard error	4.6177
Observations	466
Durbin–Watson	2.0075

	Coefficient	**Standard Error**	***t*-Statistic**
Intercept	−0.2185	0.2142	−1.0200
JPY/USD_{t-1} − JPY/USD_{t-2}	0.0287	0.0464	0.6185

Autocorrelations of the Residual			
Lag	**Autocorrelation**	**Standard Error**	***t*-Statistic**
1	−0.0023	0.0463	−0.0501
2	0.0724	0.0463	1.5643
3	0.0387	0.0463	0.8361
4	−0.0062	0.0463	−0.1329

Source: US Federal Reserve Board of Governors.

In Exhibit 19, neither the intercept nor the coefficient on the first lag of the first-differenced exchange rate differs significantly from 0, and no residual autocorrelations differ significantly from 0. These findings are consistent with the yen/US dollar exchange rate being a random walk.

We have concluded that the differenced regression is the model to choose. Now we can see that we would have been seriously misled if we had based our model choice on an R^2 comparison. In Exhibit 18, the R^2 is 0.9897, whereas in Exhibit 19, the R^2 is 0.0008. How can this be, if we just concluded that the model in Exhibit 19 is the one that we should use? In Exhibit 18, the R^2 measures how well the exchange rate in one period predicts the exchange rate in the next period. If the exchange rate is a random walk, its current value will be an extremely good predictor of its value in the next period, and thus the R^2 will be extremely high. At the same time, if the exchange rate is a random walk, then changes in the exchange rate should be completely unpredictable. Exhibit 19 estimates whether changes in the exchange rate from one month to the next can be predicted by changes in the exchange

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rate over the previous month. If they cannot be predicted, the R^2 in Exhibit 19 should be very low. In fact, it is low (0.0008). This comparison provides a good example of the general rule that we cannot necessarily choose which model is correct solely by comparing the R^2 from the two models.

The exchange rate is a random walk, and changes in a random walk are by definition unpredictable. Therefore, we cannot profit from an investment strategy that predicts changes in the exchange rate.

To this point, we have discussed only simple random walks—that is, random walks without drift. In a random walk without drift, the best predictor of the time series in the next period is its current value. A random walk with drift, however, should increase or decrease by a constant amount in each period. The equation describing a random walk with drift is a special case of the AR(1) model:

$$\begin{aligned} &x_t = b_0 + b_1 x_{t-1} + \varepsilon_t, \\ &b_1 = 1, \quad b_0 \neq 0, \text{ or} \\ &x_t = b_0 + x_{t-1} + \varepsilon_t, \quad E(\varepsilon_t) = 0. \end{aligned} \tag{9}$$

A random walk with drift has $b_0 \neq 0$, compared to a simple random walk, which has $b_0 = 0$.

We have already seen that $b_1 = 1$ implies an undefined mean-reversion level and thus nonstationarity. Consequently, we cannot use an AR model to analyze a time series that is a random walk with drift until we transform the time series by taking first differences. If we first-difference Equation 9, the result is $y_t = x_t - x_{t-1}$, $y_t = b_0 + \varepsilon_t$, $b_0 \neq 0$.

11 THE UNIT ROOT TEST OF NONSTATIONARITY

- ☐ describe implications of unit roots for time-series analysis, explain when unit roots are likely to occur and how to test for them, and demonstrate how a time series with a unit root can be transformed so it can be analyzed with an AR model
- ☐ describe the steps of the unit root test for nonstationarity and explain the relation of the test to autoregressive time-series models

In this section, we discuss how to use random walk concepts to determine whether a time series is covariance stationary. This approach focuses on the slope coefficient in the random-walk-with-drift case of an AR(1) model in contrast with the traditional autocorrelation approach, which we discuss first.

The examination of the autocorrelations of a time series at various lags is a well-known prescription for inferring whether or not a time series is stationary. Typically, for a stationary time series, either autocorrelations at all lags are statistically indistinguishable from zero or the autocorrelations drop off rapidly to zero as the number of lags becomes large. Conversely, the autocorrelations of a nonstationary time series do not exhibit those characteristics. However, this approach is less definite than a currently more popular test for nonstationarity known as the Dickey–Fuller test for a unit root.

We can explain what is known as the unit root problem in the context of an AR(1) model. If a time series comes from an AR(1) model, then to be covariance stationary, the absolute value of the lag coefficient, b_1, must be less than 1.0. We could not rely

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on the statistical results of an AR(1) model if the absolute value of the lag coefficient were greater than or equal to 1.0 because the time series would not be covariance stationary. If the lag coefficient is equal to 1.0, the time series has a **unit root**: It is a random walk and is not covariance stationary (note that when b_1 is greater than 1 in absolute value, we say that there is an "explosive root"). By definition, all random walks, with or without a drift term, have unit roots.

How do we test for unit roots in a time series? If we believed that a time series, x_t, was a random walk with drift, it would be tempting to estimate the parameters of the AR(1) model $x_t = b_0 + b_1 x_{t-1} + \varepsilon_t$ using linear regression and conduct a t-test of the hypothesis that $b_1 = 1$. Unfortunately, if $b_1 = 1$, then x_t is not covariance stationary and the t-value of the estimated coefficient, $\hat{b}_1$, does not actually follow the t-distribution; consequently, a t-test would be invalid.

Dickey and Fuller (1979) developed a regression-based unit root test based on a transformed version of the AR(1) model $x_t = b_0 + b_1 x_{t-1} + \varepsilon_t$. Subtracting x_{t-1} from both sides of the AR(1) model produces

$$x_t - x_{t-1} = b_0 + (b_1 - 1)x_{t-1} + \varepsilon_t,$$

or

$$x_t - x_{t-1} = b_0 + g_1 x_{t-1} + \varepsilon_t,\ E(\varepsilon_t) = 0, \tag{10}$$

where $g_1 = (b_1 - 1)$. If $b_1 = 1$, then $g_1 = 0$ and thus a test of $g_1 = 0$ is a test of $b_1 = 1$. If there is a unit root in the AR(1) model, then g_1 will be 0 in a regression where the dependent variable is the first difference of the time series and the independent variable is the first lag of the time series. The null hypothesis of the Dickey–Fuller test is H_0: $g_1 = 0$—that is, that the time series has a unit root and is nonstationary—and the alternative hypothesis is H_a: $g_1 < 0$, that the time series does not have a unit root and is stationary.

To conduct the test, one calculates a t-statistic in the conventional manner for $\hat{g}_1$ but instead of using conventional critical values for a t-test, one uses a revised set of values computed by Dickey and Fuller; the revised critical values are larger in absolute value than the conventional critical values. A number of software packages incorporate Dickey–Fuller tests.

EXAMPLE 11

(Historical Example)

AstraZeneca's Quarterly Sales (1)

In January 2012, equity analyst Aron Berglin is building a time-series model for the quarterly sales of AstraZeneca, a British/Swedish biopharmaceutical company headquartered in London. He is using AstraZeneca's quarterly sales in US dollars for January 2000 to December 2011 and any lagged sales data that he may need prior to 2000 to build this model. He finds that a log-linear trend model seems better suited for modeling AstraZeneca's sales than does a linear trend model. However, the Durbin–Watson statistic from the log-linear regression is just 0.7064, which causes him to reject the hypothesis that the errors in the regression are serially uncorrelated. He concludes that he cannot model the log of AstraZeneca's quarterly sales using only a time trend line. He decides to model the log of AstraZeneca's quarterly sales using an AR(1) model. He uses $\ln \text{Sales}_t = b_0 + b_1(\ln \text{Sales}_{t-1}) + \varepsilon_t$.

Before he estimates this regression, the analyst should use the Dickey–Fuller test to determine whether there is a unit root in the log of AstraZeneca's quarterly sales. If he uses the sample of quarterly data on AstraZeneca's sales from the first quarter of 2000 through the fourth quarter of 2011, takes the natural log

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of each observation, and computes the Dickey–Fuller *t*-test statistic, the value of that statistic might cause him to fail to reject the null hypothesis that there is a unit root in the log of AstraZeneca's quarterly sales.

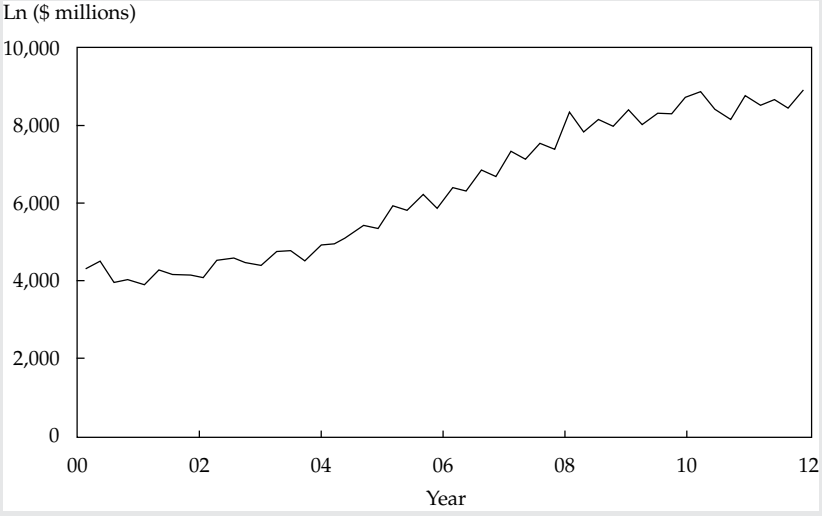
If a time series appears to have a unit root, how should we model it? One method that is often successful is to model the first-differenced series as an autoregressive time series. The following example demonstrates this method.

EXAMPLE 12

AstraZeneca's Quarterly Sales (2)

The plot of the log of AstraZeneca's quarterly sales is shown in Exhibit 20. By looking at the plot, Berglin is convinced that the log of quarterly sales is not covariance stationary (that it has a unit root).

Exhibit 20: Log of AstraZeneca's Quarterly Sales



Source: Compustat.

So he creates a new series, y_t, that is the first difference of the log of AstraZeneca's quarterly sales. Exhibit 21 shows that series.

Berglin compares Exhibit 21 to Exhibit 20 and notices that first-differencing the log of AstraZeneca's quarterly sales eliminates the strong upward trend that was present in the log of AstraZeneca's sales. Because the first-differenced series has no strong trend, Berglin is better off assuming that the differenced series is covariance stationary rather than assuming that AstraZeneca's sales or the log of AstraZeneca's sales is a covariance-stationary time series.

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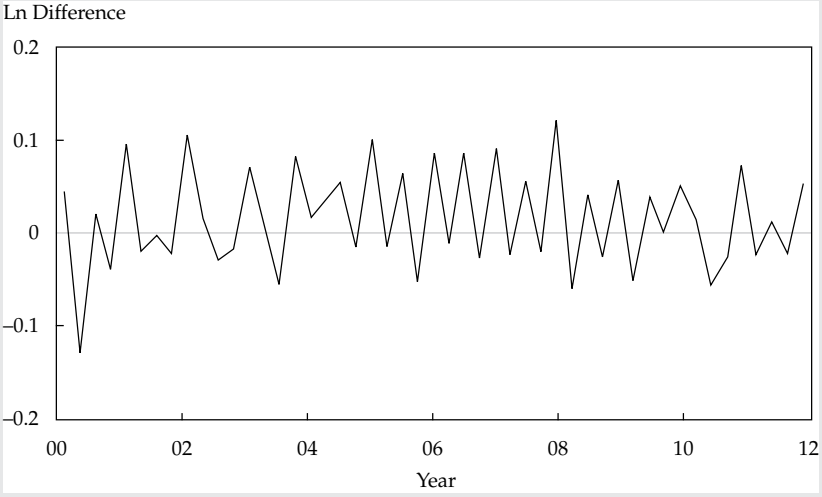
Exhibit 21: Log Difference, AstraZeneca's Quarterly Sales



Source: Compustat.

Now suppose Berglin decides to model the new series using an AR(1) model. Berglin uses $\ln(\text{Sales}_t) - \ln(\text{Sales}_{t-1}) = b_0 + b_1[\ln(\text{Sales}_{t-1}) - \ln(\text{Sales}_{t-2})] + \varepsilon_t$. Exhibit 22 shows the results of that regression.

Exhibit 22: Log Differenced Sales: AR(1) Model of AstraZeneca Quarterly Observations, January 2000–December 2011

Regression Statistics	
R^2	0.3005
Standard error	0.0475
Observations	48
Durbin–Watson	1.6874

	Coefficient	**Standard Error**	***t*-Statistic**
Intercept	0.0222	0.0071	3.1268
$\ln \text{Sales}_{t-1} - \ln \text{Sales}_{t-2}$	–0.5493	0.1236	–4.4442

Autocorrelations of the Residual

Lag	**Autocorrelation**	**Standard Error**	***t*-Statistic**
1	0.2809	0.1443	1.9466
2	–0.0466	0.1443	–0.3229
3	0.0081	0.1443	0.0561
4	0.2647	0.1443	1.8344

Source: Compustat.

The lower part of Exhibit 22 suggests that the first four autocorrelations of residuals in this model are not statistically significant. With 48 observations and two parameters, this model has 46 degrees of freedom. The critical value for a *t*-statistic in this model is above 2.0 at the 0.05 significance level. None of the *t*-statistics for these autocorrelations has an absolute value larger than 2.0.

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Therefore, we fail to reject the null hypotheses that each of these autocorrelations is equal to 0 and conclude instead that no significant autocorrelation is present in the residuals.

This result suggests that the model is well specified and that we could use the estimates. Both the intercept ($\hat{b}_0 = 0.0222$) and the coefficient ($\hat{b}_1 = -0.5493$) on the first lag of the new first-differenced series are statistically significant.

1. Explain how to interpret the estimated coefficients in the model.

Solution:

The value of the intercept (0.0222) implies that if sales have not changed in the current quarter ($y_t = \ln \text{Sales}_t - \ln \text{Sales}_{t-1} = 0$), sales will grow by 2.22% next quarter.[8] If sales have changed during this quarter, however, the model predicts that sales will grow by 2.22% minus 0.5493 times the sales growth in this quarter.

2. AstraZeneca's sales in the third and fourth quarters of 2011 were $8,405 million and $8,872 million, respectively. If we use the previous model soon after the end of the fourth quarter of 2011, what will be the predicted value of AstraZeneca's sales for the first quarter of 2012?

Solution:

Let us say that t is the fourth quarter of 2011, so $t - 1$ is the third quarter of 2011 and $t + 1$ is the first quarter of 2012. Then we would have to compute $\hat{y}_{t+1} = 0.0222 - 0.5493y_t$. To compute $\hat{y}_{t+1}$, we need to know $y_t = \ln \text{Sales}_t - \ln \text{Sales}_{t-1}$. In the third quarter of 2011, AstraZeneca's sales were $8,405 million, so $\ln \text{Sales}_{t-1} = \ln 8{,}405 = 9.0366$. In the fourth quarter of 2011, AstraZeneca's sales were $8,872 million, so $\ln \text{Sales}_t = \ln 8{,}872 = 9.0907$. Thus $y_t = 9.0907 - 9.0366 = 0.0541$. Therefore, $\hat{y}_{t+1} = 0.0222 - 0.5493(0.0541) = -0.0075$. If $\hat{y}_{t+1} = -0.0075$, then $-0.0075 = \ln \text{Sales}_{t+1} - \ln \text{Sales}_t = \ln(\text{Sales}_{t+1}/\text{Sales}_t)$. If we exponentiate both sides of this equation, the result is

$$e^{-0.0075} = \left(\frac{\text{Sales}_{t+1}}{\text{Sales}_t}\right).$$

$$\text{Sales}_{t+1} = \text{Sales}_t e^{-0.0075}$$

$$= \$8{,}872 \text{ million} \times 0.9925$$

$$= \$8{,}805 \text{ million}.$$

Thus, based on fourth quarter sales for 2011, this model would have predicted that AstraZeneca's sales in the first quarter of 2012 would be $8,805 million. This sales forecast might have affected our decision to buy AstraZeneca's stock at the time.

8 Note that 2.22 percent is the exponential growth rate, not [(Current quarter sales/Previous quarter sales) – 1]. The difference between these two methods of computing growth is usually small.

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MOVING-AVERAGE TIME-SERIES MODELS

12

So far, many of the forecasting models we have used have been autoregressive models. Because most financial time series have the qualities of an autoregressive process, autoregressive time-series models are probably the most frequently used time-series models in financial forecasting. Some financial time series, however, seem to more closely follow another kind of time-series model, called a moving-average model. For example, as we will show, returns on the S&P BSE 100 Index can be better modeled as a moving-average process than as an autoregressive process.

In this section, we present the fundamentals of moving-average models so that you can ask the right questions when considering their use. We first discuss how to smooth past values with a moving average and then how to forecast a time series using a moving-average model. Even though both methods include the words "moving average" in the name, they are very different.

Smoothing Past Values with an *n*-Period Moving Average

Suppose you are analyzing the long-term trend in the past sales of a company. In order to focus on the trend, you may find it useful to remove short-term fluctuations or noise by smoothing out the time series of sales. One technique to smooth out period-to-period fluctuations in the value of a time series is an ***n*-period moving average**. An n-period moving average of the current and past $n - 1$ values of a time series, x_t, is calculated as

$$\frac{x_t + x_{t-1} + \cdots + x_{t-(n-1)}}{n}. \tag{11}$$

The following example demonstrates how to compute a moving average of AstraZeneca's quarterly sales.

EXAMPLE 13

AstraZeneca's Quarterly Sales (3)

Suppose we want to compute the four-quarter moving average of AstraZeneca's sales as of the beginning of the first quarter of 2012. AstraZeneca's sales in the previous four quarters were as follows: 1Q 2011, $8,490 million; 2Q 2011, $8,601 million; 3Q 2011, $8,405 million; and 4Q 2011, $8,872 million. The four-quarter moving average of sales as of the beginning of the first quarter of 2012 is thus (8,490 + 8,601 + 8,405 + 8,872)/4 = $8,592 million.

We often plot the moving average of a series with large fluctuations to help discern any patterns in the data. Exhibit 23 shows monthly retail sales for the United States from December 1995 to June 2019, along with a 12-month moving average of the data (data from January 1995 are used to compute the 12-month moving average).

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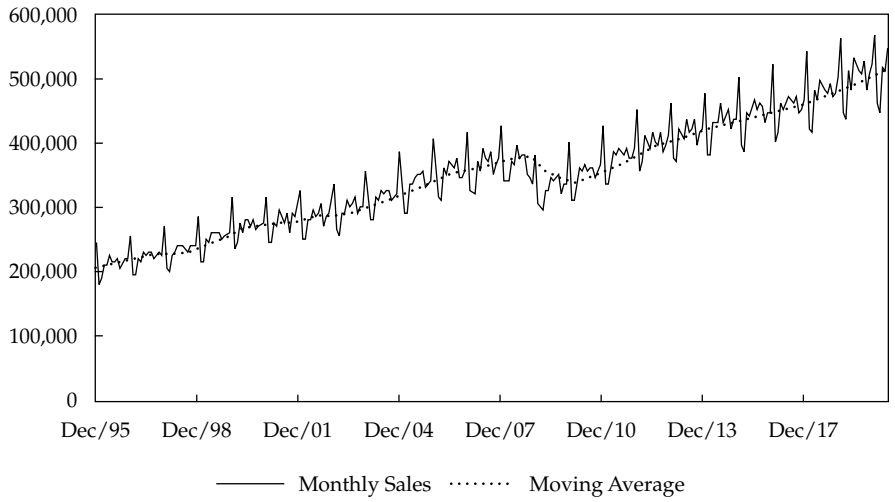
Exhibit 23: Monthly US Real Retail Sales and 12-Month Moving Average of Retail Sales



Source: Bloomberg.

As Exhibit 23 shows, each year has a very strong peak in retail sales (December) followed by a sharp drop in sales (January). Because of the extreme seasonality in the data, a 12-month moving average can help us focus on the long-term movements in retail sales instead of seasonal fluctuations. Note that the moving average does not have the sharp seasonal fluctuations of the original retail sales data. Rather, the moving average of retail sales grows steadily—for example, from 1995 through the second half of 2008—and then declines for about a year and grows steadily thereafter. We can see that trend more easily by looking at a 12-month moving average than by looking at the time series itself.

Exhibit 24 shows monthly Europe Brent Crude Oil spot prices along with a 12-month moving average of oil prices. Although these data do not have the same sharp regular seasonality displayed in the retail sales data in Exhibit 23, the moving average smooths out the monthly fluctuations in oil prices to show the longer-term movements.

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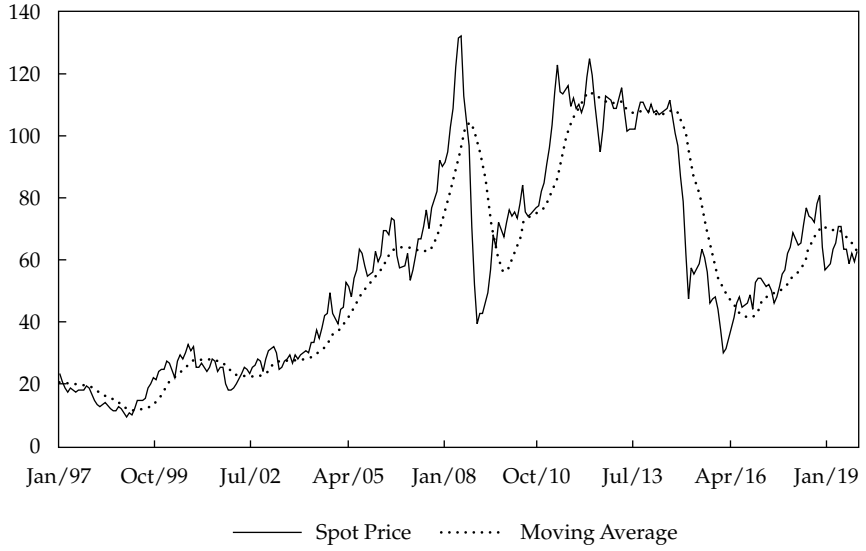
Exhibit 24: Monthly Europe Brent Crude Oil Price and 12-Month Moving Average of Prices



Source: US Energy Information Administration.

Exhibit 24 also shows one weakness with a moving average: It always lags large movements in the actual data. For example, when oil prices rose quickly in late 2007 and the first half of 2008, the moving average rose only gradually. When oil prices fell sharply toward the end of 2008, the moving average also lagged. Consequently, a simple moving average of the recent past, though often useful in smoothing out a time series, may not be the best predictor of the future. A main reason for this is that a simple moving average gives equal weight to all the periods in the moving average. In order to forecast the future values of a time series, it is often better to use a more sophisticated moving-average time-series model. We discuss such models below.

Moving-Average Time-Series Models for Forecasting

Suppose that a time series, x_t, is consistent with the following model:

$$x_t = \varepsilon_t + \theta\varepsilon_{t-1}, \quad E(\varepsilon_t) = 0, \quad E(\varepsilon_t^2) = \sigma^2,$$
$$\text{cov}(\varepsilon_t, \varepsilon_s) = E(\varepsilon_t\varepsilon_s) = 0 \text{ for } t \neq s. \tag{12}$$

This equation is called a moving-average model of order 1, or simply an MA(1) model. Theta (θ) is the parameter of the MA(1) model.[9]

Equation 12 is a moving-average model because in each period, x_t is a moving average of ε_t and ε_{t-1}, two uncorrelated random variables that each have an expected value of zero. Unlike the simple moving-average model of Equation 11, this moving-average model places different weights on the two terms in the moving average (1 on ε_t, and θ on ε_{t-1}).

9 Note that a moving-average time-series model is very different from a simple moving average, as discussed in Section 6.1. The simple moving average is based on observed values of a time series. In a moving-average time-series model, we never directly observe ε_t or any other ε_{t-j}, but we can infer how a particular moving-average model will imply a particular pattern of serial correlation for a time series, as we will discuss.

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We can see if a time series fits an MA(1) model by looking at its autocorrelations to determine whether x_t is correlated only with its preceding and following values. First, we examine the variance of x_t in Equation 12 and its first two autocorrelations. Because the expected value of x_t is 0 in all periods and ε_t is uncorrelated with its own past values, the first autocorrelation is not equal to 0, but the second and higher autocorrelations are equal to 0. Further analysis shows that all autocorrelations except for the first will be equal to 0 in an MA(1) model. Thus for an MA(1) process, any value x_t is correlated with x_{t-1} and x_{t+1} but with no other time-series values; we could say that an MA(1) model has a memory of one period.

Of course, an MA(1) model is not the most complex moving-average model. A qth-order moving-average model, denoted MA(q) and with varying weights on lagged terms, can be written as

$$\begin{aligned} &x_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}, \quad E(\varepsilon_t) = 0, \quad E(\varepsilon_t^2) = \sigma^2, \\ &\text{cov}(\varepsilon_t, \varepsilon_s) = E(\varepsilon_t \varepsilon_s) = 0 \text{ for } t \neq s. \end{aligned} \tag{13}$$

How can we tell whether an MA(q) model fits a time series? We examine the autocorrelations. For an MA(q) model, the first q autocorrelations will be significantly different from 0, and all autocorrelations beyond that will be equal to 0; an MA(q) model has a memory of q periods. This result is critical for choosing the right value of q for an MA model. We discussed this result previously for the specific case of q = 1 that all autocorrelations except for the first will be equal to 0 in an MA(1) model.

How can we distinguish an autoregressive time series from a moving-average time series? Once again, we do so by examining the autocorrelations of the time series itself. The autocorrelations of most autoregressive time series start large and decline gradually, whereas the autocorrelations of an MA(q) time series suddenly drop to 0 after the first q autocorrelations. We are unlikely to know in advance whether a time series is autoregressive or moving average. Therefore, the autocorrelations give us our best clue about how to model the time series. Most time series, however, are best modeled with an autoregressive model.

EXAMPLE 14

(Historical Example)

A Time-Series Model for Monthly Returns on the S&P BSE 100 Index

The S&P BSE 100 Index is designed to reflect the performance of India's top 100 large-cap companies listed on the BSE Ltd. (formerly Bombay Stock Exchange). Are monthly returns on the S&P BSE 100 Index autocorrelated? If so, we may be able to devise an investment strategy to exploit the autocorrelation. What is an appropriate time-series model for S&P BSE 100 monthly returns?

Exhibit 25 shows the first six autocorrelations of returns to the S&P BSE 100 using monthly data from January 2000 through December 2013. Note that all of the autocorrelations are quite small. Do they reach significance? With 168 observations, the critical value for a t-statistic in this model is about 1.98 at the 0.05 significance level. None of the autocorrelations has a t-statistic larger in absolute value than the critical value of 1.98. Consequently, we fail to reject the null hypothesis that those autocorrelations, individually, do not differ significantly from 0.

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Exhibit 25: Annualized Monthly Returns to the S&P BSE 100, January 2000–December 2013

Autocorrelations

Lag	Autocorrelation	Standard Error	*t*-Statistic
1	0.1103	0.0772	1.4288
2	–0.0045	0.0772	–0.0583
3	0.0327	0.0772	0.4236
4	0.0370	0.0772	0.4793
5	–0.0218	0.0772	–0.2824
6	0.0191	0.0772	0.2474
Observations	168		

Source: BSE Ltd.

If returns on the S&P BSE 100 were an MA(q) time series, then the first q autocorrelations would differ significantly from 0. None of the autocorrelations is statistically significant, however, so returns to the S&P BSE 100 appear to come from an MA(0) time series. An MA(0) time series in which we allow the mean to be nonzero takes the following form:[10]

$$\begin{aligned} &x_t = \mu + \varepsilon_t, \quad E(\varepsilon_t) = 0, \quad E(\varepsilon_t^2) = \sigma^2, \\ &\text{cov}(\varepsilon_t, \varepsilon_s) = E(\varepsilon_t \varepsilon_s) = 0 \text{ for } t \neq s, \end{aligned} \tag{14}$$

which means that the time series is not predictable. This result should not be surprising, because most research suggests that short-term returns to stock indexes are difficult to predict.

We can see from this example how examining the autocorrelations allowed us to choose between the AR and MA models. If returns to the S&P BSE 100 had come from an AR(1) time series, the first autocorrelation would have differed significantly from 0 and the autocorrelations would have declined gradually. Not even the first autocorrelation is significantly different from 0, however. Therefore, we can be sure that returns to the S&P BSE 100 do not come from an AR(1) model—or from any higher-order AR model, for that matter. This finding is consistent with our conclusion that the S&P BSE 100 series is MA(0).

SEASONALITY IN TIME-SERIES MODELS

13

☐ explain how to test and correct for seasonality in a time-series model and calculate and interpret a forecasted value using an AR model with a seasonal lag

10 On the basis of investment theory and evidence, we expect that the mean monthly return on the S&P BSE 100 is positive ($\mu > 0$). We can also generalize Equation 13 for an MA(q) time series by adding a constant term, μ. Including a constant term in a moving-average model does not change the expressions for the variance and autocovariances of the time series. A number of early studies of weak-form market efficiency used Equation 14 as the model for stock returns. See Garbade (1982).

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As we analyze the results of the time-series models in this reading, we encounter complications. One common complication is significant **seasonality**, a case in which the series shows regular patterns of movement within the year. At first glance, seasonality might appear to rule out using autoregressive time-series models. After all, autocorrelations will differ by season. This problem can often be solved, however, by using seasonal lags in an autoregressive model.

A seasonal lag is usually the value of the time series one year before the current period, included as an extra term in an autoregressive model. Suppose, for example, that we model a particular quarterly time series using an AR(1) model, $x_t = b_0 + b_1 x_{t-1} + \varepsilon_t$. If the time series had significant seasonality, this model would not be correctly specified. The seasonality would be easy to detect because the seasonal autocorrelation (in the case of quarterly data, the fourth autocorrelation) of the error term would differ significantly from 0. Suppose this quarterly model has significant seasonality. In this case, we might include a seasonal lag in the autoregressive model and estimate

$$x_t = b_0 + b_1 x_{t-1} + b_2 x_{t-4} + \varepsilon_t \quad (15)$$

to test whether including the seasonal lag would eliminate statistically significant autocorrelation in the error term.

In Example 15 and Example 16, we illustrate how to test and adjust for seasonality in a time-series model. We also illustrate how to compute a forecast using an autoregressive model with a seasonal lag.

EXAMPLE 15

Seasonality in Sales at Starbucks

1. Earlier, we concluded that we could not model the log of Starbucks' quarterly sales using only a time-trend line (as shown in Example 3) because the Durbin–Watson statistic from the regression provided evidence of positive serial correlation in the error term. Based on methods presented in this reading, we might next investigate using the first difference of log sales to remove an exponential trend from the data to obtain a covariance-stationary time series.

   Using quarterly data from the last quarter of 2001 to the second quarter of 2019, we estimate the following AR(1) model using ordinary least squares: $(\ln \text{Sales}_t - \ln \text{Sales}_{t-1}) = b_0 + b_1(\ln \text{Sales}_{t-1} - \ln \text{Sales}_{t-2}) + \varepsilon_t$. Exhibit 26 shows the results of the regression.

Exhibit 26: Log Differenced Sales: AR(1) Model—Starbucks, Quarterly Observations, 2001–2019

Regression Statistics	
R^2	0.2044
Standard error	0.0611

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Regression Statistics	
Observations	72
Durbin–Watson	1.9904

	Coefficient	Standard Error	*t*-Statistic
Intercept	0.0469	0.0080	5.8625
ln Sales_{t-1} – ln Sales_{t-2}	−0.4533	0.1069	−4.2404

Autocorrelations of the Residual

Lag	Autocorrelation	Standard Error	*t*-Statistic
1	0.0051	0.1179	−0.0433
2	−0.1676	0.1179	−1.4218
3	−0.0130	0.1179	−0.1099
4	0.7630	0.1179	6.4720

Source: Bloomberg.

The first thing to note in Exhibit 26 is the strong seasonal autocorrelation of the residuals. The bottom portion of the table shows that the fourth autocorrelation has a value of 0.7630 and a *t*-statistic of 6. With 72 observations and two parameters, this model has 70 degrees of freedom.[11] The critical value for a *t*-statistic is about 1.99 at the 0.05 significance level. Given this value of the *t*-statistic, we must reject the null hypothesis that the fourth autocorrelation is equal to 0 because the *t*-statistic is larger than the critical value of 1.99.

In this model, the fourth autocorrelation is the seasonal autocorrelation because this AR(1) model is estimated with quarterly data. Exhibit 26 shows the strong and statistically significant seasonal autocorrelation that occurs when a time series with strong seasonality is modeled without taking the seasonality into account. Therefore, the AR(1) model is misspecified, and we should not use it for forecasting.

Suppose we decide to use an autoregressive model with a seasonal lag because of the seasonal autocorrelation. We are modeling quarterly data, so we estimate Equation 15: $(\ln \text{Sales}_t - \ln \text{Sales}_{t-1}) = b_0 + b_1(\ln \text{Sales}_{t-1} - \ln \text{Sales}_{t-2}) + b_2(\ln \text{Sales}_{t-4} - \ln \text{Sales}_{t-5}) + \varepsilon_t$. Adding the seasonal difference $\ln \text{Sales}_{t-4} - \ln \text{Sales}_{t-5}$ is an attempt to remove a consistent quarterly pattern in the data and could also eliminate a seasonal nonstationarity if one existed. The estimates of this equation appear in Exhibit 27.

11 In this example, we restrict the start of the sample period to the beginning of 2001, and we do not use prior observations for the lags. Accordingly, the number of observations decreases with an increase in the number of lags. In Exhibit 26, the first observation is for the third quarter of 2001 because we use up to two lags. In Exhibit 27, the first observation is for the second quarter of 2002 because we use up to five lags.

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Exhibit 27: Log Differenced Sales: AR(1) Model with Seasonal Lag—Starbucks, Quarterly Observations, 2005–2019

Regression Statistics	
R^2	0.7032
Standard error	0.0373
Observations	69
Durbin–Watson	2.0392

	Coefficient	Standard Error	*t*-Statistic
Intercept	0.0107	0.0059	1.8136
$\ln \text{Sales}_{t-1} - \ln \text{Sales}_{t-2}$	−0.1540	0.0729	−2.1125
$\ln \text{Sales}_{t-4} - \ln \text{Sales}_{t-5}$	0.7549	0.0720	10.4847

Autocorrelations of the Residual			
Lag	**Autocorrelation**	**Standard Error**	***t*-Statistic**
1	0.0135	0.1204	0.1121
2	−0.0171	0.1204	−0.1420
3	0.1589	0.1204	1.3198
4	−0.1498	0.1204	−1.2442

Source: Compustat.

Note the autocorrelations of the residual shown at the bottom of Exhibit 27. None of the *t*-statistics on the first four autocorrelations is now significant. Because the overall regression is highly significant (an *F*-test, not shown in the exhibit, is significant at the 0.01 level), we can take an AR(1) model with a seasonal lag as a reasonable working model for Starbucks sales. (A model having only a seasonal lag term was investigated and not found to improve on this model.)

How can we interpret the coefficients in this model? To predict the current quarter's sales growth at Starbucks, we need to know two things: sales growth in the previous quarter and sales growth four quarters ago. If sales remained constant in each of those two quarters, the model in Exhibit 27 would predict that sales will grow by 0.0107 (1.07%) in the current quarter. If sales grew by 1% last quarter and by 2% four quarters ago, then the model would predict that sales growth this quarter will be 0.0107 − 0.0154(0.01) + 0.7549(0.02) = 0.0256, or 2.56%. Note that all of these growth rates are exponential growth rates. Notice also that the R^2 in the model with the seasonal lag (0.7032 in Exhibit 27) was more than three times higher than the R^2 in the model without the seasonal lag (0.2044 in Exhibit 26). Again, the seasonal lag model does a much better job of explaining the data.

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EXAMPLE 16

(Historical Example)

Retail Sales Growth

We want to predict the growth in monthly retail sales of Canadian furniture and home furnishing stores so that we can decide whether to recommend the shares of these stores. We decide to use non-seasonally adjusted data on retail sales. To begin with, we estimate an AR(1) model with observations on the annualized monthly growth in retail sales from January 1995 to December 2012. We estimate the following equation: Sales growth$_t$ = b_0 + b_1(Sales growth$_{t-1}$) + ε_t. Exhibit 28 shows the results from this model.

The autocorrelations of the residuals from this model, shown at the bottom of Exhibit 28, indicate that seasonality is extremely significant in this model. With 216 observations and two parameters, this model has 214 degrees of freedom. At the 0.05 significance level, the critical value for a *t*-statistic is about 1.97. The 12th-lag autocorrelation (the seasonal autocorrelation, because we are using monthly data) has a value of 0.7620 and a *t*-statistic of 11.21. The *t*-statistic on this autocorrelation is larger than the critical value (1.97), implying that we can reject the null hypothesis that the 12th autocorrelation is 0. Note also that many of the other *t*-statistics for autocorrelations shown in the table differ significantly from 0. Consequently, the model shown in Exhibit 28 is misspecified, so we cannot rely on it to forecast sales growth.

Suppose we add the seasonal lag of sales growth (the 12th lag) to the AR(1) model to estimate the equation Sales growth$_t$ = b_0 + b_1(Sales growth$_{t-1}$) + b_2(Sales growth$_{t-12}$) + ε_t. In this example, although we state that the sample period begins in 1995, we use prior observations for the lags. This results in the same number of observations irrespective of the number of lags. Exhibit 29 presents the results of estimating this equation. The estimated value of the seasonal autocorrelation (the 12th autocorrelation) has fallen to –0.1168. None of the first 12 autocorrelations has a *t*-statistic with an absolute value greater than the critical value of 1.97 at the 0.05 significance level. We can conclude that there is no significant serial correlation in the residuals from this model. Because we can reasonably believe that the model is correctly specified, we can use it to predict retail sales growth. Note that the R^2 in Exhibit 29 is 0.6724, much larger than the R^2 in Exhibit 28 (computed by the model without the seasonal lag).

Exhibit 28: Monthly Retail Sales Growth of Canadian Furniture and Home Furnishing Stores: AR(1) Model, January 1995–December 2012

Regression Statistics	
R^2	0.0509
Standard error	1.8198

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Regression Statistics	
Observations	216
Durbin–Watson	2.0956

	Coefficient	Standard Error	*t*-Statistic
Intercept	1.0518	0.1365	7.7055
Sales growth$_{t-1}$	–0.2252	0.0665	–3.3865

Autocorrelations of the Residual			
Lag	**Autocorrelation**	**Standard Error**	***t*-Statistic**
1	–0.0109	0.0680	–0.1603
2	–0.1949	0.0680	–2.8662
3	0.1173	0.0680	1.7250
4	–0.0756	0.0680	–1.1118
5	–0.1270	0.0680	–1.8676
6	–0.1384	0.0680	–2.0353
7	–0.1374	0.0680	–2.0206
8	–0.0325	0.0680	–0.4779
9	0.1207	0.0680	1.7750
10	–0.2197	0.0680	–3.2309
11	–0.0342	0.0680	–0.5029
12	0.7620	0.0680	11.2059

Source: Statistics Canada (Government of Canada).

How can we interpret the coefficients in the model? To predict growth in retail sales in this month, we need to know last month's retail sales growth and retail sales growth 12 months ago. If retail sales remained constant both last month and 12 months ago, the model in Exhibit 29 would predict that retail sales will grow at an annual rate of about 23.7% this month. If retail sales grew at an annual rate of 10% last month and at an annual rate of 5% 12 months ago, the model in Exhibit 29 would predict that retail sales will grow in the current month at an annual rate of 0.2371 – 0.0792(0.10) + 0.7798(0.05) = 0.2682, or 26.8%.

Exhibit 29: Monthly Retail Sales Growth of Canadian Furniture and Home Furnishing Stores: AR(1) Model with Seasonal Lag, January 1995–December 2012

Regression Statistics	
R^2	0.6724
Standard error	1.0717

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Regression Statistics	
Observations	216
Durbin–Watson	2.1784

	Coefficient	Standard Error	*t*-Statistic
Intercept	0.2371	0.0900	2.6344
Sales growth$_{t-1}$	−0.0792	0.0398	−1.9899
Sales growth$_{t-12}$	0.7798	0.0388	20.0979

Autocorrelations of the Residual

Lag	Autocorrelation	Standard Error	*t*-Statistic
1	−0.0770	0.0680	−1.1324
2	−0.0374	0.0680	−0.5500
3	0.0292	0.0680	0.4294
4	−0.0358	0.0680	−0.5265
5	−0.0399	0.0680	−0.5868
6	0.0227	0.0680	0.3338
7	−0.0967	0.0680	−1.4221
8	0.1241	0.0680	1.8250
9	0.0499	0.0680	0.7338
10	−0.0631	0.0680	−0.9279
11	0.0231	0.0680	0.3397
12	−0.1168	0.0680	−1.7176

Source: Statistics Canada (Government of Canada).

AR MOVING-AVERAGE MODELS AND ARCH MODELS

14

- ☐ explain autoregressive conditional heteroskedasticity (ARCH) and describe how ARCH models can be applied to predict the variance of a time series

So far, we have presented autoregressive and moving-average models as alternatives for modeling a time series. The time series we have considered in examples have usually been explained quite well with a simple autoregressive model (with or without seasonal lags).[12] Some statisticians, however, have advocated using a more general model, the autoregressive moving-average (ARMA) model. The advocates of ARMA models argue that these models may fit the data better and provide better forecasts

12 For the returns on the S&P BSE 100 (see Example 14), we chose a moving-average model over an autoregressive model.

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than do plain autoregressive (AR) models. However, as we discuss later in this section, there are severe limitations to estimating and using these models. Because you may encounter ARMA models, we next provide a brief overview.

An ARMA model combines both autoregressive lags of the dependent variable and moving-average errors. The equation for such a model with p autoregressive terms and q moving-average terms, denoted ARMA(p, q), is

$$\begin{aligned} &x_t = b_0 + b_1 x_{t-1} + \cdots + b_p x_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}, \\ &E(\varepsilon_t) = 0, \quad E(\varepsilon_t^2) = \sigma^2, \quad \text{cov}(\varepsilon_t, \varepsilon_s) = E(\varepsilon_t \varepsilon_s) = 0 \text{ for } t \neq s, \end{aligned} \tag{16}$$

where $b_1, b_2, \ldots, b_p$ are the autoregressive parameters and $\theta_1, \theta_2, \ldots, \theta_q$ are the moving-average parameters.

Estimating and using ARMA models has several limitations. First, the parameters in ARMA models can be very unstable. In particular, slight changes in the data sample or the initial guesses for the values of the ARMA parameters can result in very different final estimates of the ARMA parameters. Second, choosing the right ARMA model is more of an art than a science. The criteria for deciding on p and q for a particular time series are far from perfect. Moreover, even after a model is selected, that model may not forecast well.

To reiterate, ARMA models can be very unstable, depending on the data sample used and the particular ARMA model estimated. Therefore, you should be skeptical of claims that a particular ARMA model provides much better forecasts of a time series than any other ARMA model. In fact, in most cases, you can use an AR model to produce forecasts that are just as accurate as those from ARMA models without nearly as much complexity. Even some of the strongest advocates of ARMA models admit that these models should not be used with fewer than 80 observations, and they do not recommend using ARMA models for predicting quarterly sales or gross margins for a company using even 15 years of quarterly data.

Autoregressive Conditional Heteroskedasticity Models

Up to now, we have ignored any issues of heteroskedasticity in time-series models and have assumed homoskedasticity. **Heteroskedasticity** is the dependence of the error term variance on the independent variable; **homoskedasticity** is the independence of the error term variance from the independent variable. We have assumed that the error term's variance is constant and does not depend on the value of the time series itself or on the size of previous errors. At times, however, this assumption is violated and the variance of the error term is not constant. In such a situation, the standard errors of the regression coefficients in AR, MA, or ARMA models will be incorrect, and our hypothesis tests would be invalid. Consequently, we can make poor investment decisions based on those tests.

For example, suppose you are building an autoregressive model of a company's sales. If heteroskedasticity is present, then the standard errors of the regression coefficients of your model will be incorrect. It is likely that because of heteroskedasticity, one or more of the lagged sales terms may appear statistically significant when in fact they are not. Therefore, if you use this model for your decision making, you may make some suboptimal decisions.

In work responsible in part for his shared 2003 Nobel Prize in Economics, Robert F. Engle in 1982 first suggested a way of testing whether the variance of the error in a particular time-series model in one period depends on the variance of the error in previous periods. He called this type of heteroskedasticity "autoregressive conditional heteroskedasticity" (ARCH).

As an example, consider the ARCH(1) model

$$\varepsilon_t \sim N(0, a_0 + a_1 \varepsilon_{t-1}^2), \tag{17}$$

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where the distribution of ε_t, conditional on its value in the previous period, ε_{t-1}, is normal, with mean 0 and variance $a_0 + a_1\varepsilon_{t-1}^2$. If $a_1 = 0$, the variance of the error in every period is just a_0. The variance is constant over time and does not depend on past errors. Now suppose that $a_1 > 0$. Then the variance of the error in one period depends on how large the squared error was in the previous period. If a large error occurs in one period, the variance of the error in the next period will be even larger.

Engle showed that we can test whether a time series is ARCH(1) by regressing the squared residuals from a previously estimated time-series model (AR, MA, or ARMA) on a constant and one lag of the squared residuals. We can estimate the linear regression equation

$$\hat{\varepsilon}_t^2 = a_0 + a_1\hat{\varepsilon}_{t-1}^2 + u_t, \tag{18}$$

where u_t is an error term. If the estimate of a_1 is statistically significantly different from zero, we conclude that the time series is ARCH(1). If a time-series model has ARCH(1) errors, then the variance of the errors in period $t + 1$ can be predicted in period t using the formula $\hat{\sigma}_{t+1}^2 = \hat{a}_0 + \hat{a}_1\hat{\varepsilon}_t^2$.

EXAMPLE 17

Testing for ARCH(1) in Monthly Inflation

Analyst Lisette Miller wants to test whether monthly data on CPI inflation contain autoregressive conditional heteroskedasticity. She could estimate Equation 18 using the residuals from the time-series model. Based on the analyses in Examples 6 through 9, she has concluded that if she modeled monthly CPI inflation from 1995 to 2018, there would not be much difference in the performance of AR(1) and AR(2) models in forecasting inflation. The analyst looked at the AR(1) model for 2008–2018, found it sufficient, and decided to further explore the AR(1) model for the entire period, 1995–2018. She decides to further explore the AR(1) model for the entire period 1995 to 2018. Exhibit 30 shows the results of testing whether the errors in that model are ARCH(1). Because the test involves the first lag of residuals of the estimated time-series model, the number of observations in the test is one less than that in the model.

The *t*-statistic for the coefficient on the previous period's squared residuals is greater than 4.8. Therefore, Miller easily rejects the null hypothesis that the variance of the error does not depend on the variance of previous errors. Consequently, the test statistics she computed in Exhibit 6 are not valid, and she should not use them in deciding her investment strategy.

Exhibit 30: Test for ARCH(1) in an AR(1) Model: Residuals from Monthly CPI Inflation at an Annual Rate, March 1995–December 2018

Regression Statistics	
R^2	0.0759
Standard error	23.7841

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Regression Statistics	
Observations	286
Durbin–Watson	2.0569

	Coefficient	Standard Error	*t*-Statistic
Intercept	6.3626	1.4928	4.2622
$\hat{\varepsilon}^2_{t-1}$	0.2754	0.0570	4.8316

Source: US Bureau of Labor Statistics.

It is possible Miller's conclusion—that the AR(1) model for monthly inflation has ARCH in the errors—may have been due to the sample period used (1995–2018). In Example 9, she used a shorter sample period, 2008–2018, and concluded that monthly CPI inflation follows an AR(1) process. (These results were shown in Exhibit 16.) Exhibit 30 shows that errors for a time-series model of inflation for the entire sample (1995–2018) have ARCH errors. Do the errors estimated with a shorter sample period (2008–2018) also display ARCH? For the shorter sample period, Miller estimated an AR(1) model using monthly inflation data. Now she tests to see whether the errors display ARCH. Exhibit 31 shows the results.

In this sample, the coefficient on the previous period's squared residual has a *t*-statistic of 4.0229. Consequently, Miller rejects the null hypothesis that the errors in this regression have no autoregressive conditional heteroskedasticity. The error variance appears to be heteroskedastic, and Miller cannot rely on the *t*-statistics.

Exhibit 31: Test for ARCH(1) in an AR(1) Model: Monthly CPI Inflation at an Annual Rate, February 2008–December 2018

Regression Statistics	
R^2	0.1113
Standard error	24.64
Observations	131
Durbin–Watson	2.0385

	Coefficient	Standard Error	*t*-Statistic
Intercept	6.2082	2.2873	2.7142
$\hat{\varepsilon}^2_{t-1}$	0.3336	0.0830	4.0229

Source: US Bureau of Labor Statistics.

Suppose a model contains ARCH(1) errors. What are the consequences of that fact? First, if ARCH exists, the standard errors for the regression parameters will not be correct. We will need to use generalized least squares[13] or other methods that correct for heteroskedasticity to correctly estimate the standard error of the parameters in the time-series model. Second, if ARCH exists and we have it modeled—for example, as ARCH(1)—we can predict the variance of the errors. Suppose, for instance, that we want to predict the variance of the error in inflation using the estimated parameters from Exhibit 30: $\hat{\sigma}^2_t = 6.3626 + 0.2754\hat{\varepsilon}^2_{t-1}$. If the error in one period were 0%, the

13 See Greene (2018).

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predicted variance of the error in the next period would be 6.3626 + 0.2754(0) = 6.3626. If the error in one period were 1%, the predicted variance of the error in the next period would be 6.3626 + 0.2754(1^2) = 6.6380.

Engle and other researchers have suggested many generalizations of the ARCH(1) model, including ARCH(p) and generalized autoregressive conditional heteroskedasticity (GARCH) models. In an ARCH(p) model, the variance of the error term in the current period depends linearly on the squared errors from the previous p periods: $\sigma_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + \cdots + a_p\varepsilon_{t-p}^2$. GARCH models are similar to ARMA models of the error variance in a time series. Just like ARMA models, GARCH models can be finicky and unstable: Their results can depend greatly on the sample period and the initial guesses of the parameters in the GARCH model. Financial analysts who use GARCH models should be well aware of how delicate these models can be, and they should examine whether GARCH estimates are robust to changes in the sample and the initial guesses about the parameters.[14]

REGRESSIONS WITH MORE THAN ONE TIME SERIES 15

☐ explain how time-series variables should be analyzed for nonstationarity and/or cointegration before use in a linear regression

Up to now, we have discussed time-series models only for one time series. Although in the readings on correlation and regression and on multiple regression we used linear regression to analyze the relationship among different time series, in those readings we completely ignored unit roots. A time series that contains a unit root is not covariance stationary. If any time series in a linear regression contains a unit root, ordinary least squares estimates of regression test statistics may be invalid.

To determine whether we can use linear regression to model more than one time series, let us start with a single independent variable; that is, there are two time series, one corresponding to the dependent variable and one corresponding to the independent variable. We will then extend our discussion to multiple independent variables.

We first use a unit root test, such as the Dickey–Fuller test, for each of the two time series to determine whether either of them has a unit root.[15] There are several possible scenarios related to the outcome of these tests. One possible scenario is that we find that neither of the time series has a unit root. Then we can safely use linear regression to test the relations between the two time series. Otherwise, we may have to use additional tests, as we discuss later in this section.

EXAMPLE 18

Unit Roots and the Fisher Effect

Researchers at an asset management firm examined the Fisher effect by estimating the regression relation between expected inflation and US Treasury bill (T-bill) returns. They used 181 quarterly observations on expected inflation rates and T-bill returns from the sample period extending from the fourth quarter of 1968 through the fourth quarter of 2013. They used linear regression to analyze the relationship between the two time series. The results of this regression would

14 For more on ARCH, GARCH, and other models of time-series variance, see Hamilton (1994).
15 For theoretical details of unit root tests, see Greene (2018) or Tsay (2010). Unit root tests are available in some econometric software packages, such as EViews.

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be valid if both time series are covariance stationary; that is, neither of the two time series has a unit root. So, if they compute the Dickey–Fuller *t*-test statistic of the hypothesis of a unit root separately for each time series and find that they can reject the null hypothesis that the T-bill return series has a unit root and the null hypothesis that the expected inflation time series has a unit root, then they can use linear regression to analyze the relation between the two series. In that case, the results of their analysis of the Fisher effect would be valid.

A second possible scenario is that we reject the hypothesis of a unit root for the independent variable but fail to reject the hypothesis of a unit root for the dependent variable. In this case, the error term in the regression would not be covariance stationary. Therefore, one or more of the following linear regression assumptions would be violated: (1) that the expected value of the error term is 0, (2) that the variance of the error term is constant for all observations, and (3) that the error term is uncorrelated across observations. Consequently, the estimated regression coefficients and standard errors would be inconsistent. The regression coefficients might appear significant, but those results would be spurious.[16] Thus we should not use linear regression to analyze the relation between the two time series in this scenario.

A third possible scenario is the reverse of the second scenario: We reject the hypothesis of a unit root for the dependent variable but fail to reject the hypothesis of a unit root for the independent variable. In this case also, like the second scenario, the error term in the regression would not be covariance stationary, and we cannot use linear regression to analyze the relation between the two time series.

EXAMPLE 19

(Historical Example)

Unit Roots and Predictability of Stock Market Returns by Price-to-Earnings Ratio

Johann de Vries is analyzing the performance of the South African stock market. He examines whether the percentage change in the Johannesburg Stock Exchange (JSE) All Share Index can be predicted by the price-to-earnings ratio (P/E) for the index. Using monthly data from January 1994 to December 2013, he runs a regression using $(P_t - P_{t-1})/P_{t-1}$ as the dependent variable and P_{t-1}/E_{t-2} as the independent variable, where P_t is the value of the JSE index at time t and E_t is the earnings on the index. De Vries finds that the regression coefficient is negative and statistically significant and the value of the R^2 for the regression is quite high. What additional analysis should he perform before accepting the regression as valid?

De Vries needs to perform unit root tests for each of the two time series. If one of the two time series has a unit root, implying that it is not stationary, the results of the linear regression are not meaningful and cannot be used to conclude that stock market returns are predictable by P/E.[17]

The next possibility is that both time series have a unit root. In this case, we need to establish whether the two time series are **cointegrated** before we can rely on regression analysis.[18] Two time series are cointegrated if a long-term financial or

16 The problem of spurious regression for nonstationary time series was first discussed by Granger and Newbold (1974).

17 Barr and Kantor (1999) contains evidence that the P/E time series is nonstationary.

18 Engle and Granger (1987) first discussed cointegration.

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economic relationship exists between them such that they do not diverge from each other without bound in the long run. For example, two time series are cointegrated if they share a common trend.

In the fourth scenario, both time series have a unit root but are not cointegrated. In this scenario, as in the second and third scenarios, the error term in the linear regression will not be covariance stationary, some regression assumptions will be violated, the regression coefficients and standard errors will not be consistent, and we cannot use them for hypothesis tests. Consequently, linear regression of one variable on the other would be meaningless.

Finally, the fifth possible scenario is that both time series have a unit root but they are cointegrated. In this case, the error term in the linear regression of one time series on the other will be covariance stationary. Accordingly, the regression coefficients and standard errors will be consistent, and we can use them for hypothesis tests. However, we should be very cautious in interpreting the results of a regression with cointegrated variables. The cointegrated regression estimates the long-term relation between the two series but may not be the best model of the short-term relation between the two series. Short-term models of cointegrated series (error correction models) are discussed in Engle and Granger (1987) and Tsay (2010), but these are specialist topics.

Now let us look at how we can test for cointegration between two time series that each have a unit root, as in the fourth and fifth scenarios.[19] Engle and Granger suggested the following test. If y_t and x_t are both time series with a unit root, we should do the following:

1. Estimate the regression $y_t = b_0 + b_1 x_t + \varepsilon_t$.
2. Test whether the error term from the regression in Step 1 has a unit root using a Dickey–Fuller test. Because the residuals are based on the estimated coefficients of the regression, we cannot use the standard critical values for the Dickey–Fuller test. Instead, we must use the critical values computed by Engle and Granger, which take into account the effect of uncertainty about the regression parameters on the distribution of the Dickey–Fuller test.
3. If the (Engle–Granger) Dickey–Fuller test fails to reject the null hypothesis that the error term has a unit root, then we conclude that the error term in the regression is not covariance stationary. Therefore, the two time series are not cointegrated. In this case, any regression relation between the two series is spurious.
4. If the (Engle–Granger) Dickey–Fuller test rejects the null hypothesis that the error term has a unit root, then we may assume that the error term in the regression is covariance stationary and that the two time series are cointegrated. The parameters and standard errors from linear regression will be consistent and will let us test hypotheses about the long-term relation between the two series.

19 Consider a time series, x_t, that has a unit root. For many such financial and economic time series, the first difference of the series, $x_t - x_{t-1}$, is stationary. We say that such a series, whose first difference is stationary, has a *single* unit root. However, for some time series, even the first difference may not be stationary and further differencing may be needed to achieve stationarity. Such a time series is said to have *multiple* unit roots. In this section, we consider only the case in which each nonstationary series has a single unit root (which is quite common).

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EXAMPLE 20

Testing for Cointegration between Intel Sales and Nominal GDP

Suppose we want to test whether the natural log of Intel's sales and the natural log of GDP are cointegrated (that is, whether there is a long-term relation between GDP and Intel sales). We want to test this hypothesis using quarterly data from the first quarter of 1995 through the fourth quarter of 2019. Here are the steps:

1. Test whether the two series each have a unit root. If we cannot reject the null hypothesis of a unit root for both series, implying that both series are nonstationary, we must then test whether the two series are cointegrated.
2. Having established that each series has a unit root, we estimate the regression ln Intel sales$_t$ = b_0 + b_1(ln GDP$_t$) + ε_t, then conduct the (Engle–Granger) Dickey–Fuller test of the hypothesis that there is a unit root in the error term of this regression using the residuals from the estimated regression. If we reject the null hypothesis of a unit root in the error term of the regression, we reject the null hypothesis of no cointegration. That is, the two series would be cointegrated. If the two series are cointegrated, we can use linear regression to estimate the long-term relation between the natural log of Intel sales and the natural log of GDP.

We have so far discussed models with a single independent variable. We now extend the discussion to a model with two or more independent variables, so that there are three or more time series. The simplest possibility is that none of the time series in the model has a unit root. Then, we can safely use multiple regression to test the relation among the time series.

EXAMPLE 21

Unit Roots and Returns to the Fidelity Select Technology Fund

In earlier coverage of multiple regression, we used a multiple linear regression model to examine whether returns to either the S&P 500 Growth Index or the S&P 500 Value Index explain returns to the Fidelity Select Technology Portfolio using monthly observations between October 2015 and August 2019. Of course, if any of the three time series has a unit root, then the results of our regression analysis may be invalid. Therefore, we could use a Dickey–Fuller test to determine whether any of these series has a unit root.

If we reject the hypothesis of unit roots for all three series, we can use linear regression to analyze the relation among the series. In that case, the results of our analysis of the factors affecting returns to the Fidelity Select Technology Portfolio would be valid.

If at least one time series (the dependent variable or one of the independent variables) has a unit root while at least one time series (the dependent variable or one of the independent variables) does not, the error term in the regression cannot be covariance stationary. Consequently, we should not use multiple linear regression to analyze the relation among the time series in this scenario.

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Another possibility is that each time series, including the dependent variable and each of the independent variables, has a unit root. If this is the case, we need to establish whether the time series are cointegrated. To test for cointegration, the procedure is similar to that for a model with a single independent variable. First, estimate the regression $y_t = b_0 + b_1x_{1t} + b_2x_{2t} + \ldots + b_kx_{kt} + \varepsilon_t$. Then conduct the (Engle–Granger) Dickey–Fuller test of the hypothesis that there is a unit root in the errors of this regression using the residuals from the estimated regression.

If we cannot reject the null hypothesis of a unit root in the error term of the regression, we cannot reject the null hypothesis of no cointegration. In this scenario, the error term in the multiple regression will not be covariance stationary, so we cannot use multiple regression to analyze the relationship among the time series.

If we can reject the null hypothesis of a unit root in the error term of the regression, we can reject the null hypothesis of no cointegration. However, modeling three or more time series that are cointegrated may be difficult. For example, an analyst may want to predict a retirement services company's sales based on the country's GDP and the total population over age 65. Although the company's sales, GDP, and the population over 65 may each have a unit root and be cointegrated, modeling the cointegration of the three series may be difficult, and doing so is beyond the scope of this volume. Analysts who have not mastered all these complex issues should avoid forecasting models with multiple time series that have unit roots; the regression coefficients may be inconsistent and may produce incorrect forecasts.

OTHER ISSUES IN TIME SERIES

16

- ☐ determine an appropriate time-series model to analyze a given investment problem and justify that choice

Time-series analysis is an extensive topic and includes many highly complex issues. Our objective in this reading has been to present those issues in time series that are the most important for financial analysts and can also be handled with relative ease. In this section, we briefly discuss some of the issues that we have not covered but could be useful for analysts.

In this reading, we have shown how to use time-series models to make forecasts. We have also introduced the RMSE as a criterion for comparing forecasting models. However, we have not discussed measuring the uncertainty associated with forecasts made using time-series models. The uncertainty of these forecasts can be very large, and should be taken into account when making investment decisions. Fortunately, the same techniques apply to evaluating the uncertainty of time-series forecasts as apply to evaluating the uncertainty about forecasts from linear regression models. To accurately evaluate forecast uncertainty, we need to consider both the uncertainty about the error term and the uncertainty about the estimated parameters in the time-series model. Evaluating this uncertainty is fairly complicated when using regressions with more than one independent variable.

In this reading, we used the US CPI inflation series to illustrate some of the practical challenges analysts face in using time-series models. We used information on US Federal Reserve policy to explore the consequences of splitting the inflation series in two. In financial time-series work, we may suspect that a time series has more than one regime but lack the information to attempt to sort the data into different regimes.

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If you face such a problem, you may want to investigate other methods, especially switching regression models, to identify multiple regimes using only the time series itself.

If you are interested in these and other advanced time-series topics, you can learn more from Diebold (2008) and Tsay (2010).

Suggested Steps in Time-Series Forecasting

The following is a step-by-step guide to building a model to predict a time series.

1. Understand the investment problem you have, and make an initial choice of model. One alternative is a regression model that predicts the future behavior of a variable based on hypothesized causal relationships with other variables. Another is a time-series model that attempts to predict the future behavior of a variable based on the past behavior of the same variable.
2. If you have decided to use a time-series model, compile the time series and plot it to see whether it looks covariance stationary. The plot might show important deviations from covariance stationarity, including the following:
   - a linear trend,
   - an exponential trend,
   - seasonality, or
   - a significant shift in the time series during the sample period (for example, a change in mean or variance).
3. If you find no significant seasonality or shift in the time series, then perhaps either a linear trend or an exponential trend will be sufficient to model the time series. In that case, take the following steps:
   - Determine whether a linear or exponential trend seems most reasonable (usually by plotting the series).
   - Estimate the trend.
   - Compute the residuals.
   - Use the Durbin–Watson statistic to determine whether the residuals have significant serial correlation. If you find no significant serial correlation in the residuals, then the trend model is sufficient to capture the dynamics of the time series and you can use that model for forecasting.
4. If you find significant serial correlation in the residuals from the trend model, use a more complex model, such as an autoregressive model. First, however, reexamine whether the time series is covariance stationary. The following is a list of violations of stationarity, along with potential methods to adjust the time series to make it covariance stationary:
   - If the time series has a linear trend, first-difference the time series.
   - If the time series has an exponential trend, take the natural log of the time series and then first-difference it.
   - If the time series shifts significantly during the sample period, estimate different time-series models before and after the shift.
   - If the time series has significant seasonality, include seasonal lags (discussed in Step 7).

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5. After you have successfully transformed a raw time series into a covariance-stationary time series, you can usually model the transformed series with a short autoregression.[20] To decide which autoregressive model to use, take the following steps:
   - Estimate an AR(1) model.
   - Test to see whether the residuals from this model have significant serial correlation.
   - If you find no significant serial correlation in the residuals, you can use the AR(1) model to forecast.
6. If you find significant serial correlation in the residuals, use an AR(2) model and test for significant serial correlation of the residuals of the AR(2) model.
   - If you find no significant serial correlation, use the AR(2) model.
   - If you find significant serial correlation of the residuals, keep increasing the order of the AR model until the residual serial correlation is no longer significant.
7. Your next move is to check for seasonality. You can use one of two approaches:
   - Graph the data and check for regular seasonal patterns.
   - Examine the data to see whether the seasonal autocorrelations of the residuals from an AR model are significant (for example, the fourth autocorrelation for quarterly data) and whether the autocorrelations before and after the seasonal autocorrelations are significant. To correct for seasonality, add seasonal lags to your AR model. For example, if you are using quarterly data, you might add the fourth lag of a time series as an additional variable in an AR(1) or an AR(2) model.
8. Next, test whether the residuals have autoregressive conditional heteroskedasticity. To test for ARCH(1), for example, do the following:
   - Regress the squared residual from your time-series model on a lagged value of the squared residual.
   - Test whether the coefficient on the squared lagged residual differs significantly from 0.
   - If the coefficient on the squared lagged residual does not differ significantly from 0, the residuals do not display ARCH and you can rely on the standard errors from your time-series estimates.
   - If the coefficient on the squared lagged residual does differ significantly from 0, use generalized least squares or other methods to correct for ARCH.
9. Finally, you may also want to perform tests of the model's out-of-sample forecasting performance to see how the model's out-of-sample performance compares to its in-sample performance.

20 Most financial time series can be modeled using an autoregressive process. For a few time series, a moving-average model may fit better. To see whether this is the case, examine the first five or six autocorrelations of the time series. If the autocorrelations suddenly drop to 0 after the first q autocorrelations, a moving-average model (of order q) is appropriate. If the autocorrelations start large and decline gradually, an autoregressive model is appropriate.

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Using these steps in sequence, you can be reasonably sure that your model is correctly specified.

SUMMARY

- The predicted trend value of a time series in period t is $\hat{b}_0 + \hat{b}_1 t$ in a linear trend model; the predicted trend value of a time series in a log-linear trend model is $e^{\hat{b}_0 + \hat{b}_1 t}$.
- Time series that tend to grow by a constant amount from period to period should be modeled by linear trend models, whereas time series that tend to grow at a constant rate should be modeled by log-linear trend models.
- Trend models often do not completely capture the behavior of a time series, as indicated by serial correlation of the error term. If the Durbin–Watson statistic from a trend model differs significantly from 2, indicating serial correlation, we need to build a different kind of model.
- An autoregressive model of order p, denoted AR(p), uses p lags of a time series to predict its current value: $x_t = b_0 + b_1 x_{t-1} + b_2 x_{t-2} + \ldots + b_p x_{t-p} + \varepsilon_t$.
- A time series is covariance stationary if the following three conditions are satisfied: First, the expected value of the time series must be constant and finite in all periods. Second, the variance of the time series must be constant and finite in all periods. Third, the covariance of the time series with itself for a fixed number of periods in the past or future must be constant and finite in all periods. Inspection of a nonstationary time-series plot may reveal an upward or downward trend (nonconstant mean) and/or nonconstant variance. The use of linear regression to estimate an autoregressive time-series model is not valid unless the time series is covariance stationary.
- For a specific autoregressive model to be a good fit to the data, the autocorrelations of the error term should be 0 at all lags.
- A time series is mean reverting if it tends to fall when its level is above its long-run mean and rise when its level is below its long-run mean. If a time series is covariance stationary, then it will be mean reverting.
- The one-period-ahead forecast of a variable x_t from an AR(1) model made in period t for period $t + 1$ is $\hat{x}_{t+1} = \hat{b}_0 + \hat{b}_1 x_t$. This forecast can be used to create the two-period-ahead forecast from the model made in period t, $\hat{x}_{t+2} = \hat{b}_0 + \hat{b}_1 x_{t+1}$. Similar results hold for AR(p) models.
- In-sample forecasts are the in-sample predicted values from the estimated time-series model. Out-of-sample forecasts are the forecasts made from the estimated time-series model for a time period different from the one for which the model was estimated. Out-of-sample forecasts are usually more valuable in evaluating the forecasting performance of a time-series model than are in-sample forecasts. The root mean squared error (RMSE), defined as the square root of the average squared forecast error, is a criterion for comparing the forecast accuracy of different time-series models; a smaller RMSE implies greater forecast accuracy.

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- Just as in regression models, the coefficients in time-series models are often unstable across different sample periods. In selecting a sample period for estimating a time-series model, we should seek to assure ourselves that the time series was stationary in the sample period.
- A random walk is a time series in which the value of the series in one period is the value of the series in the previous period plus an unpredictable random error. If the time series is a random walk, it is not covariance stationary. A random walk with drift is a random walk with a nonzero intercept term. All random walks have unit roots. If a time series has a unit root, then it will not be covariance stationary.
- If a time series has a unit root, we can sometimes transform the time series into one that is covariance stationary by first-differencing the time series; we may then be able to estimate an autoregressive model for the first-differenced series.
- An n-period moving average of the current and past $(n - 1)$ values of a time series, x_t, is calculated as $[x_t + x_{t-1} + \ldots + x_{t-(n-1)}]/n$.
- A moving-average model of order q, denoted MA(q), uses q lags of a random error term to predict its current value.
- The order q of a moving-average model can be determined using the fact that if a time series is a moving-average time series of order q, its first q autocorrelations are nonzero while autocorrelations beyond the first q are zero.
- The autocorrelations of most autoregressive time series start large and decline gradually, whereas the autocorrelations of an MA(q) time series suddenly drop to 0 after the first q autocorrelations. This helps in distinguishing between autoregressive and moving-average time series.
- If the error term of a time-series model shows significant serial correlation at seasonal lags, the time series has significant seasonality. This seasonality can often be modeled by including a seasonal lag in the model, such as adding a term lagged four quarters to an AR(1) model on quarterly observations.
- The forecast made in time t for time $t + 1$ using a quarterly AR(1) model with a seasonal lag would be $x_{t+1} = \hat{b}_0 + \hat{b}_1 x_t + \hat{b}_2 x_{t-3}$.
- ARMA models have several limitations: The parameters in ARMA models can be very unstable; determining the AR and MA order of the model can be difficult; and even with their additional complexity, ARMA models may not forecast well.
- The variance of the error in a time-series model sometimes depends on the variance of previous errors, representing autoregressive conditional heteroskedasticity (ARCH). Analysts can test for first-order ARCH in a time-series model by regressing the squared residual on the squared residual from the previous period. If the coefficient on the squared residual is statistically significant, the time-series model has ARCH(1) errors.
- If a time-series model has ARCH(1) errors, then the variance of the errors in period $t + 1$ can be predicted in period t using the formula $\hat{\sigma}^2_{t+1} = \hat{a}_0 + \hat{a}_1 \hat{\varepsilon}^2_t$.
- If linear regression is used to model the relationship between two time series, a test should be performed to determine whether either time series has a unit root:
  - If neither of the time series has a unit root, then we can safely use linear regression.

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- If one of the two time series has a unit root, then we should not use linear regression.
- If both time series have a unit root and the time series are cointegrated, we may safely use linear regression; however, if they are not cointegrated, we should not use linear regression. The (Engle–Granger) Dickey–Fuller test can be used to determine whether time series are cointegrated.

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PRACTICE PROBLEMS

The following information relates to questions 1-7

Angela Martinez, an energy sector analyst at an investment bank, is concerned about the future level of oil prices and how it might affect portfolio values. She is considering whether to recommend a hedge for the bank portfolio's exposure to changes in oil prices. Martinez examines West Texas Intermediate (WTI) monthly crude oil price data, expressed in US dollars per barrel, for the 181-month period from August 2000 through August 2015. The end-of-month WTI oil price was $51.16 in July 2015 and $42.86 in August 2015 (Month 181).

After reviewing the time-series data, Martinez determines that the mean and variance of the time series of oil prices are not constant over time. She then runs the following four regressions using the WTI time-series data.

- Linear trend model: Oil price$_t = b_0 + b_1 t + e_t$.
- Log-linear trend model: ln Oil price$_t = b_0 + b_1 t + e_t$.
- AR(1) model: Oil price$_t = b_0 + b_1$Oil price$_{t-1} + e_t$.
- AR(2) model: Oil price$_t = b_0 + b_1$Oil price$_{t-1} + b_2$Oil price$_{t-2} + e_t$.

Exhibit 1 presents selected data from all four regressions, and Exhibit 2 presents selected autocorrelation data from the AR(1) models.

Exhibit 1: Crude Oil Price per Barrel, August 2000–August 2015

	Regression Statistics (*t*-statistics for coefficients are reported in parentheses)			
	Linear	**Log-Linear**	**AR(1)**	**AR(2)**
R^2	0.5703	0.6255	0.9583	0.9656
Standard error	18.6327	0.3034	5.7977	5.2799
Observations	181	181	180	179
Durbin–Watson	0.10	0.08	1.16	2.08
RMSE			2.0787	2.0530
Coefficients:				
Intercept	28.3278	3.3929	1.5948	2.0017
	(10.1846)	(74.9091)	(1.4610)	(1.9957)
t (Trend)	0.4086	0.0075		
	(15.4148)	(17.2898)		
Oil price$_{t-1}$			0.9767	1.3946
			(63.9535)	(20.2999)
Oil price$_{t-2}$				−0.4249
				(−6.2064)

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In Exhibit 1, at the 5% significance level, the lower critical value for the Durbin–Watson test statistic is 1.75 for both the linear and log-linear regressions.

Exhibit 2: Autocorrelations of the Residual from AR(1) Model

Lag	Autocorrelation	*t*-Statistic
1	0.4157	5.5768
2	0.2388	3.2045
3	0.0336	0.4512
4	−0.0426	−0.5712

Note: At the 5% significance level, the critical value for a *t*-statistic is 1.97.

After reviewing the data and regression results, Martinez draws the following conclusions.

Conclusion 1 The time series for WTI oil prices is covariance stationary.

Conclusion 2 Out-of-sample forecasting using the AR(1) model appears to be more accurate than that of the AR(2) model.

1. Based on Exhibit 1, the predicted WTI oil price for October 2015 using the linear trend model is *closest* to:

   A. $29.15.

   B. $74.77.

   C. $103.10.

2. Based on Exhibit 1, the predicted WTI oil price for September 2015 using the log-linear trend model is *closest* to:

   A. $29.75.

   B. $29.98.

   C. $116.50.

3. Based on the regression output in Exhibit 1, there is evidence of positive serial correlation in the errors in:

   A. the linear trend model but not the log-linear trend model.

   B. both the linear trend model and the log-linear trend model.

   C. neither the linear trend model nor the log-linear trend model.

4. Martinez's Conclusion 1 is:

   A. correct.

   B. incorrect because the mean and variance of WTI oil prices are not constant over time.

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C. incorrect because the Durbin–Watson statistic of the AR(2) model is greater than 1.75.

5. Based on Exhibit 1, the forecasted oil price in September 2015 based on the AR(2) model is *closest* to:

A. $38.03.

B. $40.04.

C. $61.77.

6. Based on the data for the AR(1) model in Exhibit 1 and Exhibit 2, Martinez can conclude that the:

A. residuals are not serially correlated.

B. autocorrelations do not differ significantly from zero.

C. standard error for each of the autocorrelations is 0.0745.

7. Based on the mean-reverting level implied by the AR(1) model regression output in Exhibit 1, the forecasted oil price for September 2015 is *most likely* to be:

A. less than $42.86.

B. equal to $42.86.

C. greater than $42.86.

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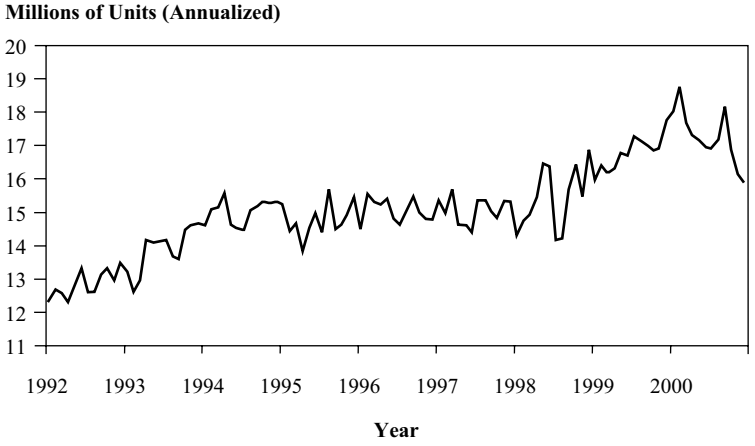
8. You have been assigned to analyze automobile manufacturers, and as a first step in your analysis, you decide to model monthly sales of lightweight vehicles to determine sales growth in that part of the industry. Exhibit 1 gives lightweight vehicle monthly sales (annualized) from January 1992 to December 2000.

Exhibit 1: Lightweight Vehicle Sales



Monthly sales in the lightweight vehicle sector, $Sales_t$, have been increasing over time, but you suspect that the growth rate of monthly sales is relatively con-

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stant. Write the simplest time-series model for Sales_t that is consistent with your perception.

The following information relates to questions 9-10

The civilian unemployment rate (UER) is an important component of many economic models. Exhibit 1 gives regression statistics from estimating a linear trend model of the unemployment rate: $\text{UER}_t = b_0 + b_1 t + \varepsilon_t$.

Exhibit 1: Estimating a Linear Trend in the Civilian Unemployment Rate: Monthly Observations, January 2013–August 2019

Regression Statistics	
R^2	0.9316
Standard error	0.3227
Observations	80
Durbin–Watson	0.1878

	Coefficient	Standard Error	*t*-Statistic
Intercept	7.2237	0.0728	99.1704
Trend	–0.0510	0.0016	–32.6136

9. Using the regression output in the previous table, what is the model's prediction of the unemployment rate for July 2013?

10. How should we interpret the Durbin–Watson (DW) statistic for this regression? What does the value of the DW statistic say about the validity of a *t*-test on the coefficient estimates?

11. Exhibit 1 compares the predicted civilian unemployment rate (PRED) with the actual civilian unemployment rate (UER) from January 2013 to August 2019. The predicted results come from estimating the linear time trend model $\text{UER}_t = b_0 + b_1 t + \varepsilon_t$.
What can we conclude about the appropriateness of this model?

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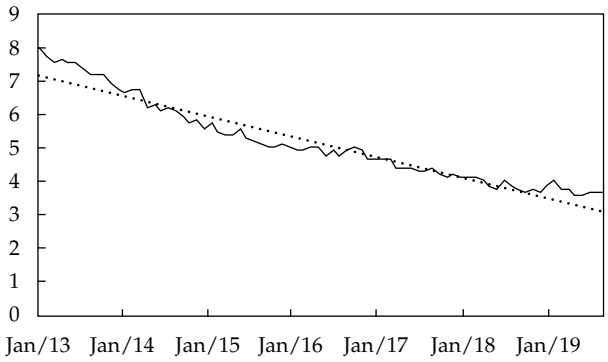
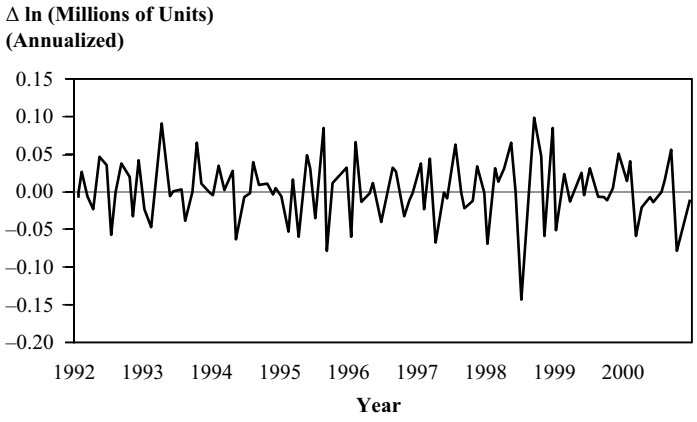
Exhibit 1: Predicted and Actual Civilian Unemployment Rates



12. Exhibit 1 shows a plot of first differences in the log of monthly lightweight vehicle sales over the same period as in Problem 11. Has differencing the data made the resulting series, $\Delta\ln(\text{Sales}_t) = \ln(\text{Sales}_t) - \ln(\text{Sales}_{t-1})$, covariance stationary?

Exhibit 1: Change in Natural Log of Lightweight Vehicle Sales



The following information relates to questions 13-14

Exhibit 1 shows a plot of the first differences in the civilian unemployment rate (UER) between January 2013 and August 2019, $\Delta\text{UER}_t = \text{UER}_t - \text{UER}_{t-1}$.

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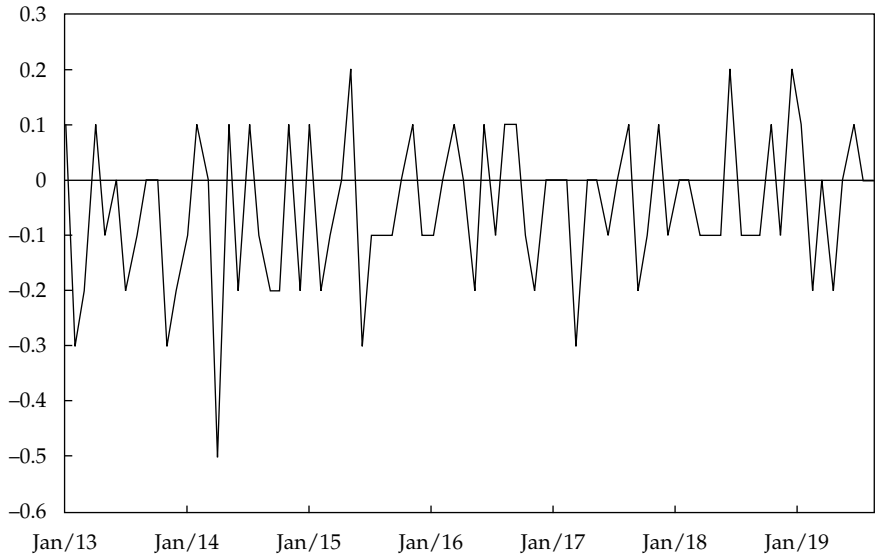
Exhibit 1: Change in Civilian Unemployment Rate



13. Has differencing the data made the new series, ΔUER_t, covariance stationary? Explain your answer.

14. Given the graph of the change in the unemployment rate shown in the figure, describe the steps we should take to determine the appropriate autoregressive time-series model specification for the series ΔUER_t.

15. Exhibit 1 gives the regression output of an AR(1) model on first differences in the unemployment rate. Describe how to interpret the DW statistic for this regression.

Exhibit 1: Estimating an AR(1) Model of Changes in the Civilian Unemployment Rate: Monthly Observations, February 2013–August 2019

Regression Statistics	
R^2	0.0546
Standard error	0.1309
Observations	79
Durbin–Watson	2.0756

	Coefficient	**Standard Error**	***t*-Statistic**
Intercept	–0.0668	0.0158	–4.2278
ΔUER_{t-1}	–0.2320	0.1100	–2.191

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The following information relates to questions 16-17

Using monthly data from January 1992 to December 2000, we estimate the following equation for lightweight vehicle sales: $\Delta\ln(\text{Sales}_t) = 2.7108 + 0.3987\Delta\ln(\text{Sales}_{t-1}) + \varepsilon_t$. Exhibit 1 gives sample autocorrelations of the errors from this model.

Exhibit 1: Different Order Autocorrelations of Differences in the Logs of Vehicle Sales

Lag	Autocorrelation	Standard Error	*t*-Statistic
1	0.9358	0.0962	9.7247
2	0.8565	0.0962	8.9005
3	0.8083	0.0962	8.4001
4	0.7723	0.0962	8.0257
5	0.7476	0.0962	7.7696
6	0.7326	0.0962	7.6137
7	0.6941	0.0962	7.2138
8	0.6353	0.0962	6.6025
9	0.5867	0.0962	6.0968
10	0.5378	0.0962	5.5892
11	0.4745	0.0962	4.9315
12	0.4217	0.0962	4.3827

16. Use the information in the table to assess the appropriateness of the specification given by the equation.

17. If the residuals from the AR(1) model above violate a regression assumption, how would you modify the AR(1) specification?

18. Assume that changes in the civilian unemployment rate are covariance stationary and that an AR(1) model is a good description for the time series of changes in the unemployment rate. Specifically, we have $\Delta\text{UER}_t = -0.0668 - 0.2320\Delta\text{UER}_{t-1}$ (using the coefficient estimates given in the previous problem). Given this equation, what is the mean-reverting level to which changes in the unemployment rate converge?

The following information relates to questions 19-21

Suppose the following model describes changes in the civilian unemployment rate: $\Delta\text{UER}_t = -0.0668 - 0.2320\Delta\text{UER}_{t-1}$. The current change (first difference) in the unemployment rate is 0.0300. Assume that the mean-reverting level for changes in the unemployment rate is –0.0542.

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19. What is the best prediction of the next change?

20. What is the prediction of the change following the next change?

21. Explain your answer to Part B in terms of equilibrium.

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22. Exhibit 1 gives the actual sales, log of sales, and changes in the log of sales of Cisco Systems for the period 1Q 2019 to 4Q 2019.

Exhibit 1

Date	Actual Sales ($ Millions)	Log of Sales	Changes in Log of Sales $\Delta\ln(Sales_t)$
1Q 2019	13,072	9.4782	0.0176
2Q 2019	12,446	9.4292	–0.0491
3Q 2019	12,958	9.4695	0.0403
4Q 2019	13,428	9.5051	0.0356
1Q 2020			
2Q 2020			

Forecast the first- and second-quarter sales of Cisco Systems for 2020 using the regression $\Delta\ln(Sales_t) = 0.0068 + 0.2633\Delta\ln(Sales_{t-1})$.

The following information relates to questions 23-24

Exhibit 1 gives the actual change in the log of sales of Cisco Systems from 1Q 2019 to 4Q 2019, along with the forecasts from the regression model $\Delta\ln(Sales_t) = 0.0068 + 0.2633\Delta\ln(Sales_{t-1})$ estimated using data from 1Q 2001 to 4Q 2018. (Note that the observations after the fourth quarter of 2018 are out of sample.)

Exhibit 1

Date	Actual Value of Changes in the Log of Sales $\Delta\ln(Sales_t)$	Forecast Value of Changes in the Log of Sales $\Delta\ln(Sales_t)$
1Q 2019	0.0176	0.0147
2Q 2019	–0.0491	0.0107
3Q 2019	0.0403	0.0096
4Q 2019	0.0356	0.0093

23. Calculate the RMSE for the out-of-sample forecast errors.

24. Compare the forecasting performance of the model given with that of another model having an out-of-sample RMSE of 2%.

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The following information relates to questions 25-26

25. The AR(1) model for the civilian unemployment rate, $\Delta UER_t = -0.0405 - 0.4674\Delta UER_{t-1}$, was developed with five years of data. What would be the drawback to using the AR(1) model to predict changes in the civilian unemployment rate 12 months or more ahead, as compared with 1 month ahead?

26. For purposes of estimating a predictive equation, what would be the drawback to using 30 years of civilian unemployment data rather than only 5 years?

The following information relates to questions 27-35

Max Busse is an analyst in the research department of a large hedge fund. He was recently asked to develop a model to predict the future exchange rate between two currencies. Busse gathers monthly exchange rate data from the most recent 10-year period and runs a regression based on the following AR(1) model specification:

Regression 1: $x_t = b_0 + b_1 x_{t-1} + \varepsilon_t$, where x_t is the exchange rate at time t.

Based on his analysis of the time series and the regression results, Busse reaches the following conclusions:

Conclusion 1 The variance of x_t increases over time.

Conclusion 2 The mean-reverting level is undefined.

Conclusion 3 b_0 does not appear to be significantly different from 0.

Busse decides to do additional analysis by first-differencing the data and running a new regression.

Regression 2: $y_t = b_0 + b_1 y_{t-1} + \varepsilon_t$, where $y_t = x_t - x_{t-1}$.

Exhibit 1 shows the regression results.

Exhibit 1: First-Differenced Exchange Rate AR(1) Model: Month-End Observations, Last 10 Years

Regression Statistics	
R^2	0.0017
Standard error	7.3336

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Regression Statistics	
Observations	118
Durbin–Watson	1.9937

	Coefficient	Standard Error	*t*-Statistic
Intercept	−0.8803	0.6792	−1.2960
$x_{t-1} - x_{t-2}$	0.0412	0.0915	0.4504

Autocorrelations of the Residual

Lag	Autocorrelation	Standard Error	*t*-Statistic
1	0.0028	0.0921	0.0300
2	0.0205	0.0921	0.2223
3	0.0707	0.0921	0.7684
4	0.0485	0.0921	0.5271

Note: The critical *t*-statistic at the 5% significance level is 1.98.

Busse decides that he will need to test the data for nonstationarity using a Dickey–Fuller test. To do so, he knows he must model a transformed version of Regression 1.

Busse's next assignment is to develop a model to predict future quarterly sales for PoweredUP, Inc., a major electronics retailer. He begins by running the following regression:

Regression 3: $\ln \text{Sales}_t - \ln \text{Sales}_{t-1} = b_0 + b_1(\ln \text{Sales}_{t-1} - \ln \text{Sales}_{t-2}) + \varepsilon_t$.

Exhibit 2 presents the results of this regression.

Exhibit 2: Log Differenced Sales AR(1) Model: PoweredUP, Inc., Last 10 Years of Quarterly Sales

Regression Statistics	
R^2	0.2011
Standard error	0.0651
Observations	38
Durbin–Watson	1.9677

	Coefficient	Standard Error	*t*-Statistic
Intercept	0.0408	0.0112	3.6406
$\ln \text{Sales}_{t-1} - \ln \text{Sales}_{t-2}$	−0.4311	0.1432	−3.0099

Autocorrelations of the Residual

Lag	Autocorrelation	Standard Error	*t*-Statistic
1	0.0146	0.1622	0.0903
2	−0.1317	0.1622	−0.8119
3	−0.1123	0.1622	−0.6922
4	0.6994	0.1622	4.3111

Note: The critical *t*-statistic at the 5% significance level is 2.02.

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Because the regression output from Exhibit 2 raises some concerns, Busse runs a different regression. These regression results, along with quarterly sales data for the past five quarters, are presented in Exhibit 3 and Exhibit 4, respectively.

Exhibit 3: Log Differenced Sales AR(1) Model with Seasonal Lag: PoweredUP, Inc., Last 10 Years of Quarterly Sales

Regression Statistics

R^2	0.6788
Standard error	0.0424
Observations	35
Durbin–Watson	1.8799

	Coefficient	Standard Error	*t*-Statistic
Intercept	0.0092	0.0087	1.0582
$\ln \text{Sales}_{t-1} - \ln \text{Sales}_{t-2}$	−0.1279	0.1137	−1.1252
$\ln \text{Sales}_{t-4} - \ln \text{Sales}_{t-5}$	0.7239	0.1093	6.6209

Autocorrelations of the Residual

Lag	Autocorrelation	Standard Error	*t*-Statistic
1	0.0574	0.1690	0.3396
2	0.0440	0.1690	0.2604
3	0.1923	0.1690	1.1379
4	−0.1054	0.1690	−0.6237

Note: The critical *t*-statistic at the 5% significance level is 2.03.

Exhibit 4: Most Recent Quarterly Sales Data (in billions)

Dec 2015 (Sales_{t-1})	$3.868
Sep 2015 (Sales_{t-2})	$3.780
June 2015 (Sales_{t-3})	$3.692
Mar 2015 (Sales_{t-4})	$3.836
Dec 2014 (Sales_{t-5})	$3.418

After completing his work on PoweredUP, Busse is asked to analyze the relationship of oil prices and the stock prices of three transportation companies. His firm wants to know whether the stock prices can be predicted by the price of oil. Exhibit 5 shows selected information from the results of his analysis.

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Exhibit 5: Analysis Summary of Stock Prices for Three Transportation Stocks and the Price of Oil

	Unit Root?	Linear or Exponential Trend?	Serial Correlation of Residuals in Trend Model?	ARCH(1)?	Comments
Company 1	Yes	Exponential	Yes	Yes	Not cointegrated with oil price
Company 2	Yes	Linear	Yes	No	Cointegrated with oil price
Company 3	No	Exponential	Yes	No	Not cointegrated with oil price
Oil Price	Yes				

To assess the relationship between oil prices and stock prices, Busse runs three regressions using the time series of each company's stock prices as the dependent variable and the time series of oil prices as the independent variable.

27. Which of Busse's conclusions regarding the exchange rate time series is consistent with both the properties of a covariance-stationary time series and the properties of a random walk?

A. Conclusion 1

B. Conclusion 2

C. Conclusion 3

28. Based on the regression output in Exhibit 1, the first-differenced series used to run Regression 2 is consistent with:

A. a random walk.

B. covariance stationarity.

C. a random walk with drift.

29. Based on the regression results in Exhibit 1, the *original* time series of exchange rates:

A. has a unit root.

B. exhibits stationarity.

C. can be modeled using linear regression.

30. In order to perform the nonstationarity test, Busse should transform the Regression 1 equation by:

A. adding the second lag to the equation.

B. changing the regression's independent variable.

C. subtracting the independent variable from both sides of the equation.

31. Based on the regression output in Exhibit 2, what should lead Busse to conclude that the Regression 3 equation is not correctly specified?

A. The Durbin–Watson statistic

B. The *t*-statistic for the slope coefficient

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C. The *t*-statistics for the autocorrelations of the residual

32. Based on the regression output in Exhibit 3 and sales data in Exhibit 4, the forecasted value of quarterly sales for March 2016 for PoweredUP is *closest* to:

A. $4.193 billion.

B. $4.205 billion.

C. $4.231 billion.

33. Based on Exhibit 5, Busse should conclude that the variance of the error terms for Company 1:

A. is constant.

B. can be predicted.

C. is homoskedastic.

34. Based on Exhibit 5, for which company would the regression of stock prices on oil prices be expected to yield valid coefficients that could be used to estimate the long-term relationship between stock price and oil price?

A. Company 1

B. Company 2

C. Company 3

35. Based on Exhibit 5, which single time-series model would *most likely* be appropriate for Busse to use in predicting the future stock price of Company 3?

A. Log-linear trend model

B. First-differenced AR(2) model

C. First-differenced log AR(1) model

The following information relates to questions 36-37

Exhibit 1 shows monthly observations on the natural log of lightweight vehicle sales, $\ln(\text{Sales}_t)$, for January 1992 to December 2000.

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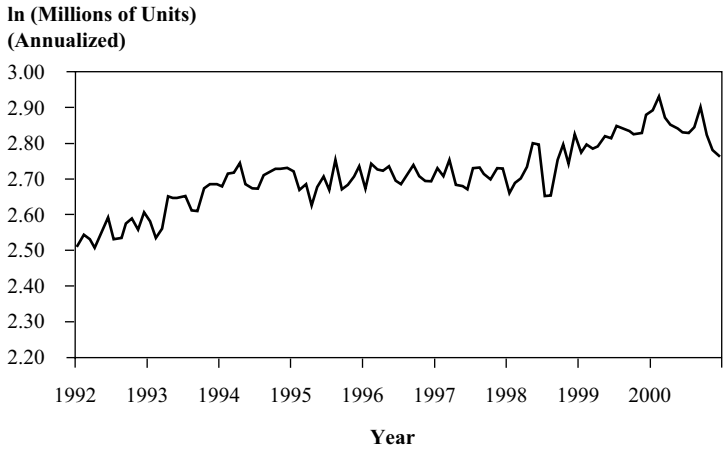
Exhibit 1: Lightweight Vehicle Sales



36. Using the figure, comment on whether the specification $\ln(\text{Sales}_t) = b_0 + b_1[\ln(\text{Sales}_{t-1})] + \varepsilon_t$ is appropriate.

37. State an appropriate transformation of the time series.

38. Suppose we want to predict the annualized return of the five-year T-bill using the annualized return of the three-month T-bill with monthly observations from January 1993 to December 2002. Our analysis produces the data shown in Exhibit 1.

Exhibit 1: Regression with Three-Month T-Bill as the Independent Variable and the Five-Year T-Bill as the Dependent Variable: Monthly Observations, January 1993–December 2002

Regression Statistics	
R^2	0.5829
Standard error	0.6598
Observations	120
Durbin–Watson	0.1130

	Coefficient	Standard Error	*t*-Statistic
Intercept	3.0530	0.2060	14.8181
Three-month	0.5722	0.0446	12.8408

Can we rely on the regression model in Exhibit 1 to produce meaningful predictions? Specify what problem might be a concern with this regression.

39. Exhibit 1 shows the quarterly sales of Avon Products from 1Q 1992 to 2Q 2002.

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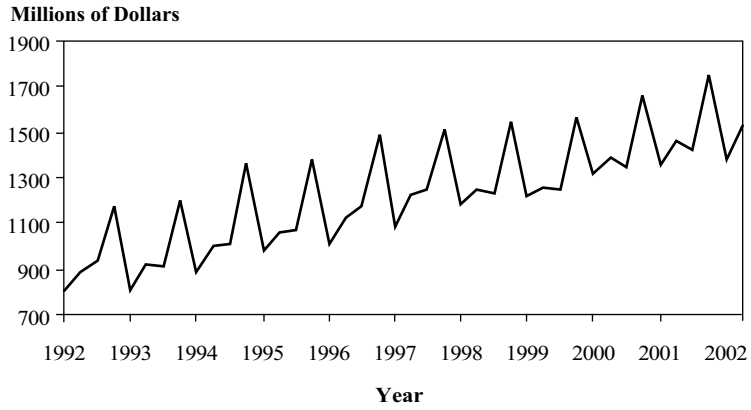
Describe the salient features of the data shown.

Exhibit 1: Quarterly Sales at Avon



The following information relates to questions 40-41

Exhibit 1 shows the autocorrelations of the residuals from an AR(1) model fit to the changes in the gross profit margin (GPM) of the Home Depot, Inc.

Exhibit 1: Autocorrelations of the Residuals from Estimating the Regression $\Delta GPM_t = 0.0006 - 0.3330\Delta GPM_{t-1} + \varepsilon_t$, 1Q 1992–4Q 2001 (40 Observations)

Lag	Autocorrelation
1	−0.1106
2	−0.5981
3	−0.1525
4	0.8496
5	−0.1099

Exhibit 2 shows the output from a regression on changes in the GPM for Home Depot, where we have changed the specification of the AR regression.

Exhibit 2: Change in Gross Profit Margin for Home Depot, 1Q 1992–4Q 2001

Regression Statistics	
R^2	0.9155
Standard error	0.0057

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Regression Statistics	
Observations	40
Durbin–Watson	2.6464

	Coefficient	Standard Error	*t*-Statistic
Intercept	–0.0001	0.0009	–0.0610
ΔGPM_{t-1}	–0.0608	0.0687	–0.8850
ΔGPM_{t-4}	0.8720	0.0678	12.8683

40. Identify the change that was made to the regression model.

41. Discuss the rationale for changing the regression specification.

The following information relates to questions 42-43

Suppose we decide to use an autoregressive model with a seasonal lag because of the seasonal autocorrelation in the previous problem. We are modeling quarterly data, so we estimate Equation 15: $(\ln \text{Sales}_t - \ln \text{Sales}_{t-1}) = b_0 + b_1(\ln \text{Sales}_{t-1} - \ln \text{Sales}_{t-2}) + b_2(\ln \text{Sales}_{t-4} - \ln \text{Sales}_{t-5}) + \varepsilon_t$. Exhibit 1 shows the regression statistics from this equation.

Exhibit 1: Log Differenced Sales: AR(1) Model with Seasonal Lag Johnson & Johnson Quarterly Observations, January 1985–December 2001

Regression Statistics	
R^2	0.4220
Standard error	0.0318
Observations	68
Durbin–Watson	1.8784

	Coefficient	Standard Error	*t*-Statistic
Intercept	0.0121	0.0053	2.3055
Lag 1	–0.0839	0.0958	–0.8757
Lag 4	0.6292	0.0958	6.5693

Autocorrelations of the Residual

Lag	Autocorrelation	Standard Error	*t*-Statistic
1	0.0572	0.1213	0.4720
2	–0.0700	0.1213	–0.5771
3	0.0065	0.1213	–0.0532
4	–0.0368	0.1213	–0.3033

42. Using the information in Exhibit 1, determine whether the model is correctly

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specified.

43. If sales grew by 1% last quarter and by 2% four quarters ago, use the model to predict the sales growth for this quarter.

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44. Describe how to test for autoregressive conditional heteroskedasticity (ARCH) in the residuals from the AR(1) regression on first differences in the civilian unemployment rate, $\Delta UER_t = b_0 + b_1 \Delta UER_{t-1} + \varepsilon_t$.

The following information relates to questions 45-47

Exhibit 1 shows the quarterly sales of Cisco Systems from 3Q 2001 to 2Q 2019.

Exhibit 1: Quarterly Sales at Cisco

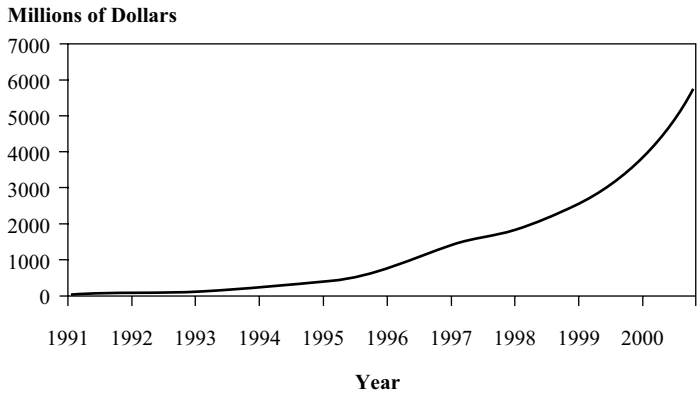

Exhibit 2 gives the regression statistics from estimating the model $\Delta \ln(Sales_t) = b_0 + b_1 \Delta \ln(Sales_{t-1}) + \varepsilon_t$.

Exhibit 2: Change in the Natural Log of Sales for Cisco Quarterly Observations, 3Q 1991–4Q 2000

Regression Statistics	
R^2	0.2899
Standard error	0.0408
Observations	38
Durbin–Watson	1.5707

	Coefficient	Standard Error	*t*-Statistic
Intercept	0.0661	0.0175	3.7840
$\Delta \ln(Sales_{t-1})$	0.4698	0.1225	3.8339

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45. Describe the salient features of the quarterly sales series.

46. Describe the procedures we should use to determine whether the AR(1) specification is correct.

47. Assuming the model is correctly specified, what is the long-run change in the log of sales toward which the series will tend to converge?

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SOLUTIONS

1. C is correct. The predicted value for period t from a linear trend is calculated as $\hat{y}_t = \hat{b}_0 + \hat{b}_1(t)$.
   October 2015 is the second month out of sample, or $t = 183$. So, the predicted value for October 2015 is calculated as

   $\hat{y}_t = 28.3278 + 0.4086(183) = \$103.10.$

   Therefore, the predicted WTI oil price for October 2015 based on the linear trend model is $103.10.

2. C is correct. The predicted value for period t from a log-linear trend is calculated as $\ln \hat{y}_t = \hat{b}_0 + \hat{b}_1(t)$.
   September 2015 is the first month out of sample, or $t = 182$. So, the predicted value for September 2015 is calculated as follows:

   $\ln \hat{y}_t = 3.3929 + 0.0075(182).$

   $\ln \hat{y}_t = 4.7579.$

   $\hat{y}_t = e^{4.7579} = \$116.50.$

   Therefore, the predicted WTI oil price for September 2015, based on the log-linear trend model, is $116.50.

3. B is correct. The Durbin–Watson statistic for the linear trend model is 0.10 and for the log-linear trend model is 0.08. Both of these values are below the critical value of 1.75. Therefore, we can reject the hypothesis of no positive serial correlation in the regression errors in both the linear trend model and the log-linear trend model.

4. B is correct. There are three requirements for a time series to be covariance stationary. First, the expected value of the time series must be constant and finite in all periods. Second, the variance of the time series must be constant and finite in all periods. Third, the covariance of the time series with itself for a fixed number of periods in the past or future must be constant and finite in all periods. Martinez concludes that the mean and variance of the time series of WTI oil prices are not constant over time. Therefore, the time series is not covariance stationary.

5. B is correct. The last two observations in the WTI time series are July and August 2015, when the WTI oil price was $51.16 and $42.86, respectively. Therefore, September 2015 represents a one-period-ahead forecast. The one-period-ahead forecast from an AR(2) model is calculated as

   $\hat{x}_{t+1} = \hat{b}_0 + \hat{b}_1 x_t + \hat{b}_2 x_{t-1}.$

   So, the one-period-ahead (September 2015) forecast is calculated as

   $\hat{x}_{t+1} = 2.0017 + 1.3946(\$42.86) - 0.4249(\$51.16) = \$40.04.$

   Therefore, the September 2015 forecast based on the AR(2) model is $40.04.

6. C is correct. The standard error of the autocorrelations is calculated as $\frac{1}{\sqrt{T}}$, where T represents the number of observations used in the regression. Therefore, the standard error for each of the autocorrelations is $\frac{1}{\sqrt{180}} = 0.0745$. Martinez can conclude that the residuals are serially correlated and are significantly different

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from zero because two of the four autocorrelations in Exhibit 2 have a *t*-statistic in absolute value that is greater than the critical value of 1.97.

Choices A and B are incorrect because two of the four autocorrelations have a *t*-statistic in absolute value that is greater than the critical value of the *t*-statistic of 1.97.

7. C is correct. The mean-reverting level from the AR(1) model is calculated as

$$\hat{x}_t = \frac{b_0}{1 - b_1} = \frac{1.5948}{1 - 0.9767} = \$68.45.$$

Therefore, the mean-reverting WTI oil price from the AR(1) model is $68.45. The forecasted oil price in September 2015 will likely be greater than $42.86 because the model predicts that the price will rise in the next period from the August 2015 price of $42.86.

8. A log-linear model captures growth at a constant rate. The log-linear model $\ln(\text{Sales}_t) = b_0 + b_1 t + \varepsilon_t$ would be the simplest model consistent with a constant growth rate for monthly sales. Note that we would need to confirm that the regression assumptions are satisfied before accepting the model as valid.

9. The estimated forecasting equation is $\text{UER}_t = 5.5098 - 0.0294(t)$. The data begin in January 2013, and July 2013 is Period 7. Thus the linear trend model predicts the unemployment rate to be $\text{UER}_7 = 7.2237 - 0.0510(7) = 6.8667$, or approximately 6.9%.

10. The DW statistic is designed to detect positive serial correlation of the errors of a regression equation. Under the null hypothesis of no positive serial correlation, the DW statistic is 2.0. Positive serial correlation will lead to a DW statistic that is less than 2.0. From the table in Problem 1, we see that the DW statistic is 0.1878. To see whether this result is significantly less than 2.0, refer to the Durbin–Watson table in Appendix E at the end of this volume, in the column marked k = 1 (one independent variable) and the row corresponding to 80 observations. We see that $d_l = 1.55$. Because our DW statistic is clearly less than d_l, we reject the null hypothesis of no serial correlation at the 0.05 significance level.

    The presence of serial correlation in the error term violates one of the regression assumptions. The standard errors of the estimated coefficients will be biased downward, so we cannot conduct hypothesis testing on the coefficients.

11. The difference between UER and its forecast value, PRED, is the forecast error. In an appropriately specified regression model, the forecast errors are randomly distributed around the regression line and have a constant variance. We can see that the errors from this model specification are persistent. The errors tend first to be above the regression line, and then, starting in 2014, they tend to be below the regression line until 2017, when they again are persistently above the regression line. This persistence suggests that the errors are positively serially correlated. Therefore, we conclude that the model is not appropriate for making estimates.

12. The plot of the series $\Delta\ln(\text{Sales}_t)$ appears to fluctuate around a constant mean; its volatility seems constant throughout the period. Differencing the data appears to have made the time series covariance stationary.

13. The plot of the series ΔUER_t seems to fluctuate around a constant mean; its volatility appears to be constant throughout the period. Our initial judgment is that the differenced series is covariance stationary.

14. The change in the unemployment rate seems covariance stationary, so we should first estimate an AR(1) model and test to see whether the residuals from this

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model have significant serial correlation. If the residuals do not display significant serial correlation, we should use the AR(1) model. If the residuals do display significant serial correlation, we should try an AR(2) model and test for serial correlation of the residuals of the AR(2) model. We should continue this procedure until the errors from the final AR(p) model are serially uncorrelated.

15. The DW statistic cannot be appropriately used for a regression that has a lagged value of the dependent variable as one of the explanatory variables. To test for serial correlation, we need to examine the autocorrelations.

16. In a correctly specified regression, the residuals must be serially uncorrelated. We have 108 observations, so the standard error of the autocorrelation is $1/\sqrt{T}$, or in this case $1/\sqrt{108} = 0.0962$. The t-statistic for each lag is significant at the 0.01 level. We would have to modify the model specification before continuing with the analysis.

17. Because the residuals from the AR(1) specification display significant serial correlation, we should estimate an AR(2) model and test for serial correlation of the residuals of the AR(2) model. If the residuals from the AR(2) model are serially uncorrelated, we should then test for seasonality and ARCH behavior. If any serial correlation remains in the residuals, we should estimate an AR(3) process and test the residuals from that specification for serial correlation. We should continue this procedure until the errors from the final AR(p) model are serially uncorrelated. When serial correlation is eliminated, we should test for seasonality and ARCH behavior.

18. When a covariance-stationary series is at its mean-reverting level, the series will tend not to change until it receives a shock (ε_t). So, if the series ΔUER_t is at the mean-reverting level, $\Delta UER_t = \Delta UER_{t-1}$. This implies that $\Delta UER_t = -0.0668 - 0.2320\Delta UER_t$, so that $(1 + 0.2320)\Delta UER_t = -0.0668$ and $\Delta UER_t = -0.0668/(1 + 0.2320) = -0.0542$. The mean-reverting level is −0.0542. In an AR(1) model, the general expression for the mean-reverting level is $b_0/(1 - b_1)$.

19. The predicted change in the unemployment rate for next period is −7.38%, found by substituting 0.0300 into the forecasting model: $-0.0668 - 0.2320(0.03) = -0.0738$.

20. If we substitute our one-period-ahead forecast of −0.0738 into the model (using the chain rule of forecasting), we get a two-period-ahead forecast of −0.0497, or −4.97%.

21. The answer to Part B is quite close to the mean-reverting level of −0.0542. A stationary time series may need many periods to return to its equilibrium, mean-reverting level.

22. The forecast of sales is $13,647 million for the first quarter of 2020 and $13,800 million for the second quarter of 2002, as the following table shows.

Date	Sales ($ Millions)	Log of Sales	Actual Value of Changes in the Log of Sales $\Delta\ln(Sales_t)$	Forecast Value of Changes in the Log of Sales $\Delta\ln(Sales_t)$
1Q 2019	13,072	9.4782	0.0176	
2Q 2019	12,446	9.4292	−0.0491	
3Q 2019	12,958	9.4695	0.0403	
4Q 2019	13,428	9.5051	0.0356	

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Date	Sales ($ Millions)	Log of Sales	Actual Value of Changes in the Log of Sales $\Delta\ln(\text{Sales}_t)$	Forecast Value of Changes in the Log of Sales $\Delta\ln(\text{Sales}_t)$
1Q 2020	13,647	9.5213		0.0162
2Q 2020	13,800	9.5324		0.0111

We find the forecasted change in the log of sales for the first quarter of 2020 by inputting the value for the change in the log of sales from the previous quarter into the equation $\Delta\ln(\text{Sales}_t) = 0.0068 + 0.2633\Delta\ln(\text{Sales}_{t-1})$. Specifically, $\Delta\ln(\text{Sales}_t) = 0.0068 + 0.2633(0.0356) = 0.0162$, which means that we forecast the log of sales in the first quarter of 2020 to be 9.5051 + 0.0162 = 9.5213.

Next, we forecast the change in the log of sales for the second quarter of 2020 as $\Delta\ln(\text{Sales}_t) = 0.0068 + 0.2633(0.0162) = 0.0111$. Note that we have to use our first-quarter 2020 estimated value of the change in the log of sales as our input for $\Delta\ln(\text{Sales}_{t-1})$ because we are forecasting past the period for which we have actual data.

With a forecasted change of 0.0111, we forecast the log of sales in the second quarter of 2020 to be 9.5213 + 0.0111 = 9.5324.

We have forecasted the log of sales in the first and second quarters of 2020 to be 9.5213 and 9.5324, respectively. Finally, we take the antilog of our estimates of the log of sales in the first and second quarters of 2020 to get our estimates of the level of sales: $e^{9.5213} = 13{,}647$ and $e^{9.5324} = 13{,}800$, respectively, for sales of $13,647 million and $13,800 million.

23. The RMSE of the out-of-sample forecast errors is approximately 3.6%.
Out-of-sample error refers to the difference between the realized value and the forecasted value of $\Delta\ln(\text{Sales}_t)$ for dates beyond the estimation period. In this case, the out-of-sample period is 1Q 2019 to 4Q 2019. These are the four quarters for which we have data that we did not use to obtain the estimated model $\Delta\ln(\text{Sales}_t) = 0.0068 + 0.2633\Delta\ln(\text{Sales}_{t-1})$.

The steps to calculate RMSE are as follows:

i. Take the difference between the actual and the forecast values. This is the error.
ii. Square the error.
iii. Sum the squared errors.
iv. Divide by the number of forecasts.
v. Take the square root of the average.

We show the calculations for RMSE in the following table.

Actual Values of Changes in the Log of Sales $\Delta\ln(\text{Sales}_t)$	Forecast Values of Changes in the Log of Sales $\Delta\ln(\text{Sales}_t)$	Error (Column 1 – Column 2)	Squared Error (Column 3 Squared)
0.0176	0.0147	0.0029	0.0000
–0.0491	0.0107	–0.0598	0.0036
0.0403	0.0096	0.0307	0.0009
0.0356	0.0093	0.0263	0.0007
		Sum	0.0052
		Mean	0.0013
		RMSE	0.036

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24. The lower the RMSE, the more accurate the forecasts of a model in forecasting. Therefore, the model with the RMSE of 2% has greater accuracy in forecasting than the model in Part A, which has an RMSE of 3.6%.

25. Predictions too far ahead can be nonsensical. For example, the AR(1) model we have been examining, $\Delta UER_t = -0.0405 - 0.4674\Delta UER_{t-1}$, taken at face value, predicts declining civilian unemployment into the indefinite future. Because the civilian unemployment rate will probably not go below 3% frictional unemployment and cannot go below 0% unemployment, this model's long-range forecasts are implausible. The model is designed for short-term forecasting, as are many time-series models.

26. Using more years of data for estimation may lead to nonstationarity even in the series of first differences in the civilian unemployment rate. As we go further back in time, we increase the risk that the underlying civilian unemployment rate series has more than one regime (or true model). If the series has more than one regime, fitting one model to the entire period would not be correct. Note that when we have good reason to believe that a time series is stationary, a longer series of data is generally desirable.

27. C is correct. A random walk can be described by the equation $x_t = b_0 + b_1 x_{t-1} + \varepsilon_t$, where $b_0 = 0$ and $b_1 = 1$. So $b_0 = 0$ is a characteristic of a random walk time series. A covariance-stationary series must satisfy the following three requirements:

    1. The expected value of the time series must be constant and finite in all periods.
    2. The variance of the time series must be constant and finite in all periods.
    3. The covariance of the time series with itself for a fixed number of periods in the past or future must be constant and finite in all periods.

    $b_0 = 0$ does not violate any of these three requirements and is thus consistent with the properties of a covariance-stationary time series.

28. B is correct. The critical *t*-statistic at a 5% confidence level is 1.98. As a result, neither the intercept nor the coefficient on the first lag of the first-differenced exchange rate in Regression 2 differs significantly from zero. Also, the residual autocorrelations do not differ significantly from zero. As a result, Regression 2 can be reduced to $y_t = \varepsilon_t$, with a mean-reverting level of $b_0/(1 - b_1) = 0/1 = 0$. Therefore, the variance of y_t in each period is $\text{var}(\varepsilon_t) = \sigma^2$. The fact that the residuals are not autocorrelated is consistent with the covariance of the times series with itself being constant and finite at different lags. Because the variance and the mean of $y_t$are constant and finite in each period, we can also conclude that $y_t$is covariance stationary.

29. A is correct. If the exchange rate series is a random walk, then the first-differenced series will yield $b_0 = 0$ and $b_1 = 0$ and the error terms will not be serially correlated. The data in Exhibit 1 show that this is the case: Neither the intercept nor the coefficient on the first lag of the first-differenced exchange rate in Regression 2 differs significantly from zero because the *t*-statistics of both coefficients are less than the critical *t*-statistic of 1.98. Also, the residual autocorrelations do not differ significantly from zero because the *t*-statistics of all autocorrelations are less than the critical *t*-statistic of 1.98. Therefore, because all random walks have unit roots, the exchange rate time series used to run Regression 1 has a unit root.

30. C is correct. To conduct the Dickey–Fuller test, one must subtract the independent variable, x_{t-1},from both sides of the original AR(1) model. This results

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in a change of the dependent variable (from x_t to $x_t - x_{t-1}$) and a change in the regression's slope coefficient (from b_1 to $b_1 - 1$) but not a change in the independent variable.

31. C is correct. The regression output in Exhibit 2 suggests there is serial correlation in the residual errors. The fourth autocorrelation of the residual has a value of 0.6994 and a *t*-statistic of 4.3111, which is greater than the *t*-statistic critical value of 2.02. Therefore, the null hypothesis that the fourth autocorrelation is equal to zero can be rejected. This indicates strong and significant seasonal autocorrelation, which means the Regression 3 equation is misspecified.

32. C is correct. The quarterly sales for March 2016 are calculated as follows:

$$\ln \text{Sales}_t - \ln \text{Sales}_{t-1} = b_0 + b_1(\ln \text{Sales}_{t-1} - \ln \text{Sales}_{t-2}) + b_2(\ln \text{Sales}_{t-4} - \ln \text{Sales}_{t-5}).$$

$$\ln \text{Sales}_t - \ln 3.868 = 0.0092 - 0.1279(\ln 3.868 - \ln 3.780) + 0.7239(\ln 3.836 - \ln 3.418).$$

$$\ln \text{Sales}_t - 1.35274 = 0.0092 - 0.1279(1.35274 - 1.32972) + 0.7239(1.34443 - 1.22906).$$

$$\ln \text{Sales}_t = 1.35274 + 0.0092 - 0.1279(0.02301) + 0.7239(0.11538).$$

$$\ln \text{Sales}_t = 1.44251.$$

$$\text{Sales}_t = e^{1.44251} = 4.231.$$

33. B is correct. Exhibit 3 shows that the time series of the stock prices of Company 1 exhibits heteroskedasticity, as evidenced by the fact that the time series is ARCH(1). If a time series is ARCH(1), then the variance of the error in one period depends on the variance of the error in previous periods. Therefore, the variance of the errors in period $t + 1$ can be predicted in period t using the formula

$$\hat{\sigma}^2_{t+1} = \hat{a}_0 + \hat{a}_1 \hat{\varepsilon}^2_t.$$

34. B is correct. When two time series have a unit root but are cointegrated, the error term in the linear regression of one time series on the other will be covariance stationary. Exhibit 5 shows that the series of stock prices of Company 2 and the oil prices both contain a unit root and the two time series are cointegrated. As a result, the regression coefficients and standard errors are consistent and can be used for hypothesis tests. Although the cointegrated regression estimates the long-term relation between the two series, it may not be the best model of the short-term relationship.

35. C is correct. As a result of the exponential trend in the time series of stock prices for Company 3, Busse would want to take the natural log of the series and then first-difference it. Because the time series also has serial correlation in the residuals from the trend model, Busse should use a more complex model, such as an autoregressive (AR) model.

36. The graph of $\ln(\text{Sales}_t)$ appears to trend upward over time. A series that trends upward or downward over time often has a unit root and is thus not covariance stationary. Therefore, using an AR(1) regression on the undifferenced series is probably not correct. In practice, we need to examine regression statistics to confirm such visual impressions.

37. The most common way to transform a time series with a unit root into a

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covariance-stationary time series is to difference the data—that is, to create a new series: $\Delta\ln(\text{Sales}_t) = \ln(\text{Sales}_t) - \ln(\text{Sales}_{t-1})$.

38. To determine whether we can use linear regression to model more than one time series, we should first determine whether any of the time series has a unit root. If none of the time series has a unit root, then we can safely use linear regression to test the relations between the two time series. Note that if one of the two variables has a unit root, then our analysis would not provide valid results; if both of the variables have unit roots, then we would need to evaluate whether the variables are cointegrated.

39. The quarterly sales of Avon show an upward trend and a clear seasonal pattern, as indicated by the repeated regular cycle.

40. A second explanatory variable, the change in the gross profit margin lagged four quarters, ΔGPM_{t-4}, was added.

41. The model was augmented to account for seasonality in the time series (with quarterly data, significant autocorrelation at the fourth lag indicates seasonality). The standard error of the autocorrelation coefficient equals 1 divided by the square root of the number of observations: $1/\sqrt{40}$, or 0.1581. The autocorrelation at the fourth lag (0.8496) is significant: $t = 0.8496/0.1581 = 5.37$. This indicates seasonality, and accordingly we added ΔGPM_{t-4}. Note that in the augmented regression, the coefficient on ΔGPM_{t-4} is highly significant. (Although the autocorrelation at second lag is also significant, the fourth lag is more important because of the rationale of seasonality. Once the fourth lag is introduced as an independent variable, we might expect that the second lag in the residuals would not be significant.)

42. In order to determine whether this model is correctly specified, we need to test for serial correlation among the residuals. We want to test whether we can reject the null hypothesis that the value of each autocorrelation is 0 against the alternative hypothesis that each is not equal to 0. At the 0.05 significance level, with 68 observations and three parameters, this model has 65 degrees of freedom. The critical value of the *t*-statistic needed to reject the null hypothesis is thus about 2.0. The absolute value of the *t*-statistic for each autocorrelation is below 0.60 (less than 2.0), so we cannot reject the null hypothesis that each autocorrelation is not significantly different from 0. We have determined that the model is correctly specified.

43. If sales grew by 1% last quarter and by 2% four quarters ago, then the model predicts that sales growth this quarter will be $0.0121 - 0.0839[\ln(1.01)] + 0.6292[\ln(1.02)] = e^{0.02372} - 1 = 2.40\%$.

44. We should estimate the regression $\Delta\text{UER}_t = b_0 + b_1\Delta\text{UER}_{t-1} + \varepsilon_t$ and save the residuals from the regression. Then we should create a new variable, $\hat{\varepsilon}_t^2$, by squaring the residuals. Finally, we should estimate $\hat{\varepsilon}_t^2 = a_0 + a_1\hat{\varepsilon}_{t-1}^2 + u_t$ and test to see whether a_1 is statistically different from 0.

45. The series has a steady upward trend of growth, suggesting an exponential growth rate. This finding suggests transforming the series by taking the natural log and differencing the data.

46. First, we should determine whether the residuals from the AR(1) specification are serially uncorrelated. If the residuals are serially correlated, then we should try an AR(2) specification and then test the residuals from the AR(2) model for serial correlation. We should continue in this fashion until the residuals are se-

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rially uncorrelated and then look for seasonality in the residuals. If seasonality is present, we should add a seasonal lag. If no seasonality is present, we should test for ARCH. If ARCH is not present, we can conclude that the model is correctly specified.

47. If the model $\Delta\ln(\text{Sales}_t) = b_0 + b_1[\Delta\ln(\text{Sales}_{t-1})] + \varepsilon_t$ is correctly specified, then the series $\Delta\ln(\text{Sales}_t)$ is covariance stationary. So, this series tends to its mean-reverting level, which is $b_0/(1 - b_1)$, or $0.0661/(1 - 0.4698) = 0.1247$.

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LEARNING MODULE

6

Machine Learning

by Kathleen DeRose, CFA, Matthew Dixon, PhD, FRM, and Christophe Le Lannou.

Kathleen DeRose, CFA, is at New York University, Stern School of Business (USA). Matthew Dixon, PhD, FRM, is at Illinois Institute of Technology, Stuart School of Business (USA). Christophe Le Lannou is at dataLearning (United Kingdom).

LEARNING OUTCOMES

Mastery	*The candidate should be able to:*
☐	describe supervised machine learning, unsupervised machine learning, and deep learning
☐	describe overfitting and identify methods of addressing it
☐	describe supervised machine learning algorithms—including penalized regression, support vector machine, k-nearest neighbor, classification and regression tree, ensemble learning, and random forest—and determine the problems for which they are best suited
☐	describe unsupervised machine learning algorithms—including principal components analysis, k-means clustering, and hierarchical clustering—and determine the problems for which they are best suited
☐	describe neural networks, deep learning nets, and reinforcement learning

INTRODUCTION 1

☐	describe supervised machine learning, unsupervised machine learning, and deep learning

Investment firms are increasingly using technology at every step of the investment management value chain—from improving their understanding of clients to uncovering new sources of alpha and executing trades more efficiently. Machine learning techniques, a central part of that technology, are the subject of this reading. These techniques first appeared in finance in the 1990s and have since flourished with the explosion of data and cheap computing power.

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This reading provides a high-level view of machine learning (ML). It covers a selection of key ML algorithms and their investment applications. Investment practitioners should be equipped with a basic understanding of the types of investment problems that machine learning can address, an idea of how the algorithms work, and the vocabulary to interact with machine learning and data science experts. While investment practitioners need not master the details and mathematics of machine learning, as domain experts in investments they can play an important role in the implementation of these techniques by being able to source appropriate model inputs, interpret model outputs, and translate outputs into appropriate investment actions.

Section 2 gives an overview of machine learning in investment management. Section 3 defines machine learning and the types of problems that can be addressed by supervised and unsupervised learning. Section 4 describes evaluating machine learning algorithm performance. Key supervised machine learning algorithms are covered in Section 5, and Section 6 describes key unsupervised machine learning algorithms. Neural networks, deep learning nets, and reinforcement learning are covered in Section 7. Section 8 provides a decision flowchart for selecting the appropriate ML algorithm. The reading concludes with a summary.

Machine Learning and Investment Management

The growing volume and exploding diversity of data, as well as the perceived increasing economic value of insights extracted from these data, have inspired rapid growth in data science. This newly emerging field combines mathematics, computer science, and business analytics. It also strikes out in a new direction that relies on learning—from basic learning functions that map relationships between variables to advanced neural networks that mimic physical processes that absorb, order, and adapt to information.

Machine learning has theoretical and practical implications for investment management. For example, machine learning could potentially reshape accepted wisdom about asset risk premiums and reconfigure investment management business processes. Large datasets and learning models are already affecting investment management practices—from client profiling to asset allocation, stock selection, portfolio construction and risk management, and trading.

Machine learning applications are at each step of the asset and wealth management value chain. Chatbots answer basic retirement savings questions, learning from their interactions with investors. Machine learning methods can be used to generate alpha signals used in security selection by creating a non-linear forecast for a single time series, by deriving a forecast from a suite of predefined factors, or even by choosing input signals from existing or newly found data. For example, researchers using textual analysis have found that year-over-year changes in annual (10-K) and quarterly (10-Q) filings, particularly negative changes in the management discussion and risk sections, can strongly predict equity returns.

Machine learning methods can help calculate target portfolio weights that incorporate client restrictions and then dynamically weight them to maximize a Sharpe ratio. Another use of machine learning methods is better estimation of the variance–covariance matrix via principal components analysis, which reduces the number of variables needed to explain the variation in the data. Research suggests that machine learning solutions outperform mean–variance optimization in portfolio construction. Machine learning techniques are already creating better order flow management tools with non-linear trading algorithms that reduce the costs of implementing portfolio decisions. These developments have caused an evolution in the automation of tools, processes, and businesses (such as robo-advising).

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WHAT IS MACHINE LEARNING

2

- ☐ describe supervised machine learning, unsupervised machine learning, and deep learning

We now discuss some fundamental concepts of machine learning, including a definition and an overview of key types of machine learning, such as supervised and unsupervised ML.

Defining Machine Learning

Statistical approaches and machine learning techniques both analyze observations to reveal some underlying process; however, they diverge in their assumptions, terminology, and techniques. Statistical approaches rely on foundational assumptions and explicit models of structure, such as observed samples that are assumed to be drawn from a specified underlying probability distribution. These a priori restrictive assumptions can fail in reality.

In contrast, machine learning seeks to extract knowledge from large amounts of data with fewer such restrictions. The goal of machine learning algorithms is to automate decision-making processes by generalizing (i.e., "learning") from known examples to determine an underlying structure in the data. The emphasis is on the ability of the algorithm to generate structure or predictions from data without any human help. An elementary way to think of ML algorithms is to "find the pattern, apply the pattern."

Machine learning techniques are better able than statistical approaches (such as linear regression) to handle problems with many variables (high dimensionality) or with a high degree of non-linearity. ML algorithms are particularly good at detecting change, even in highly non-linear systems, because they can detect the preconditions of a model's break or anticipate the probability of a regime switch.

Machine learning is broadly divided into three distinct classes of techniques: supervised learning, unsupervised learning, and deep learning/reinforcement learning.

Supervised Learning

Supervised learning involves ML algorithms that infer patterns between a set of inputs (the *X*'s) and the desired output (*Y*). The inferred pattern is then used to map a given input set into a predicted output. Supervised learning requires a **labeled dataset**, one that contains matched sets of observed inputs and the associated output. Applying the ML algorithm to this dataset to infer the pattern between the inputs and output is called "training" the algorithm. Once the algorithm has been trained, the inferred pattern can be used to predict output values based on new inputs (i.e., ones not in the training dataset).

Multiple regression is an example of supervised learning. A regression model takes matched data (*X*'s, *Y*) and uses it to estimate parameters that characterize the relationship between *Y* and the *X*'s. The estimated parameters can then be used to predict *Y* on a new, different set of *X*'s. The difference between the predicted and actual *Y* is used to evaluate how well the regression model predicts out-of-sample (i.e., using new data).

The terminology used with ML algorithms differs from that used in regression. Exhibit 1 provides a visual of the supervised learning model training process and a translation between regression and ML terminologies.

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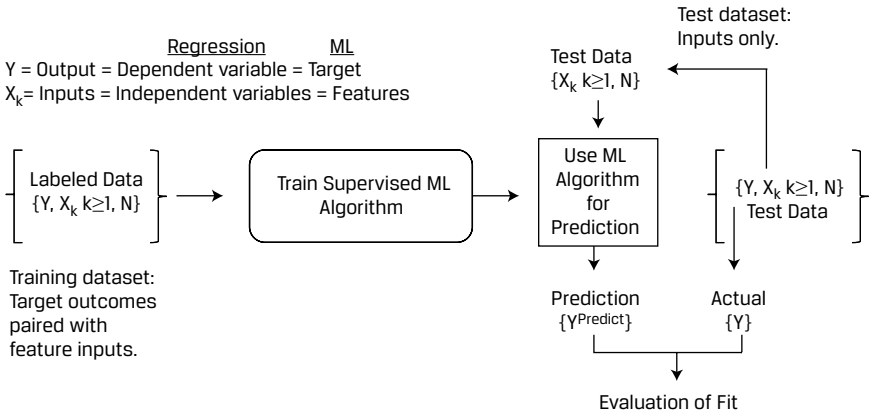
Exhibit 1: Overview of Supervised Learning



In supervised machine learning, the dependent variable (Y) is the **target** and the independent variables (X's) are known as **features**. The labeled data (training dataset) is used to train the supervised ML algorithm to infer a pattern-based prediction rule. The fit of the ML model is evaluated using labeled test data in which the predicted targets ($Y^{Predict}$) are compared to the actual targets (Y^{Actual}).

An example of supervised learning is the case in which ML algorithms are used to predict whether credit card transactions are fraudulent or legitimate. In the credit card example, the target is a binary variable with a value of 1 for "fraudulent" or 0 for "non-fraudulent." The features are the transaction characteristics. The chosen ML algorithm uses these data elements to train a model to predict the likelihood of fraud more accurately in new transactions. The ML program "learns from experience" if the percentage of correctly predicted credit card transactions increases as the amount of input from a growing credit card database increases. One possible ML algorithm to use would be to fit a logistic regression model to the data to provide an estimate of the probability a transaction is fraudulent.

Supervised learning can be divided into two categories of problems—regression and classification—with the distinction between them being determined by the nature of the target (Y) variable. If the target variable is continuous, then the task is one of regression (even if the ML technique used is not "regression"; note this nuance of ML terminology). If the target variable is categorical or ordinal (i.e., a ranked category), then it is a classification problem. Regression and classification use different ML techniques.

Regression focuses on making predictions of continuous target variables. Most readers are already familiar with multiple linear regression (e.g., ordinary least squares) models, but other supervised learning techniques exist, including non-linear models. These non-linear models are useful for problems involving large datasets with large numbers of features, many of which may be correlated. Some examples of problems belonging to the regression category are using historical stock market returns to forecast stock price performance or using historical corporate financial ratios to forecast the probability of bond default.

Classification focuses on sorting observations into distinct categories. In (supervised) machine learning, when the dependent variable (target) is categorical, the model relating the outcome to the independent variables (features) is called a "classifier." You should already be familiar with logistic regression as a type of classifier. Many classification models are binary classifiers, as in the case of fraud detection for credit card transactions. Multi-category classification is not uncommon, as in the case of

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classifying firms into multiple credit rating categories. In assigning ratings, the outcome variable is ordinal, meaning the categories have a distinct order or ranking (e.g., from low to high creditworthiness). Ordinal variables are intermediate between categorical variables and continuous variables on a scale of measurement.

Unsupervised Learning

Unsupervised learning is machine learning that does not make use of labeled data. More formally, in unsupervised learning, we have inputs (*X*'s) that are used for analysis without any target (*Y*) being supplied. In unsupervised learning, because the ML algorithm is not given labeled training data, the algorithm seeks to discover structure within the data themselves. As such, unsupervised learning is useful for exploring new datasets because it can provide human experts with insights into a dataset too big or too complex to visualize.

Two important types of problems that are well suited to unsupervised machine learning are reducing the dimension of data and sorting data into clusters, known as dimension reduction and clustering, respectively.

Dimension reduction focuses on reducing the number of features while retaining variation across observations to preserve the information contained in that variation. Dimension reduction may have several purposes. It may be applied to data with a large number of features to produce a lower dimensional representation (i.e., with fewer features) that can fit, for example, on a computer screen. Dimension reduction is also used in many quantitative investment and risk management applications where it is critical to identify the most predictive factors underlying asset price movements.

Clustering focuses on sorting observations into groups (clusters) such that observations in the same cluster are more similar to each other than they are to observations in other clusters. Groups are formed based on a set of criteria that may or may not be prespecified (such as the number of groups). Clustering has been used by asset managers to sort companies into groupings driven by data (e.g., based on their financial statement data or corporate characteristics) rather than conventional groupings (e.g., based on sectors or countries).

Deep Learning and Reinforcement Learning

More broadly in the field of artificial intelligence, additional categories of machine learning algorithms are distinguished. In **deep learning**, sophisticated algorithms address complex tasks, such as image classification, face recognition, speech recognition, and natural language processing. Deep learning is based on **neural networks** (NNs), also called artificial neural networks (ANNs)—highly flexible ML algorithms that have been successfully applied to a variety of supervised and unsupervised tasks characterized by large datasets, non-linearities, and interactions among features. In **reinforcement learning**, a computer learns from interacting with itself or data generated by the same algorithm. Deep learning and reinforcement learning principles have been combined to create efficient algorithms for solving a range of highly complex problems in robotics, health care, and finance.

Summary of ML Algorithms and How to Choose among Them

Exhibit 2 is a guide to the various machine learning algorithms organized by algorithm type (supervised or unsupervised) and by type of variables (continuous, categorical, or both). We will not cover linear or logistic regression since they are covered elsewhere in readings on quantitative methods. The extensions of linear regression, such as penalized regression and least absolute shrinkage and selection operator (LASSO), as well as the other ML algorithms shown in Exhibit 2, will be covered in this reading.

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Exhibit 2: Guide to ML Algorithms

	ML Algorithm Type	
Variables	**Supervised (Target Variable)**	**Unsupervised (No Target Variable)**
Continuous	**Regression**	**Dimension Reduction**
	• Linear; Penalized Regression/LASSO	• Principal Components Analysis (PCA)
	• Logistic	**Clustering**
	• Classification and Regression Tree (CART)	• *K*-Means
	• Random Forest	• Hierarchical
Categorical	**Classification**	**Dimension Reduction**
	• Logistic	• Principal Components Analysis (PCA)
	• Support Vector Machine (SVM)	**Clustering**
	• *K*-Nearest Neighbor (KNN)	• *K*-Means
	• Classification and Regression Tree (CART)	• Hierarchical
Continuous or Categorical	Neural Networks	Neural Networks
	Deep Learning	Deep Learning
	Reinforcement Learning	Reinforcement Learning

EXAMPLE 1

Machine Learning Overview

1. Which of the following *best* describes machine learning? Machine learning:

   A. is a type of computer algorithm used just for linear regression.

   B. is a set of algorithmic approaches aimed at generating structure or predictions from data without human intervention by finding a pattern and then applying the pattern.

   C. is a set of computer-driven approaches adapted to extracting information from linear, labeled datasets.

Solution:

B is correct. A is incorrect because machine learning algorithms are typically not used for linear regression. C is incorrect because machine learning is not limited to extracting information from linear, labeled datasets.

2. Which of the following statements is *most* accurate? When attempting to discover groupings of data without any target (*Y*) variable:

   A. an unsupervised ML algorithm is used.

   B. an ML algorithm that is given labeled training data is used.

   C. a supervised ML algorithm is used.

Solution:

A is correct. B is incorrect because the term "labeled training data" means the target (*Y*) is provided. C is incorrect because a supervised ML algorithm is meant to predict a target (*Y*) variable.

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3. Which of the following statements concerning supervised learning *best* distinguishes it from unsupervised learning? Supervised learning involves:

   A. training on labeled data to infer a pattern-based prediction rule.

   B. training on unlabeled data to infer a pattern-based prediction rule.

   C. learning from unlabeled data by discovering underlying structure in the data themselves.

Solution:

A is correct. B is incorrect because supervised learning uses labeled training data. C is incorrect because it describes unsupervised learning.

4. Which of the following *best* describes dimension reduction? Dimension reduction:

   A. focuses on classifying observations in a dataset into known groups using labeled training data.

   B. focuses on clustering observations in a dataset into unknown groups using unlabeled data.

   C. focuses on reducing the number of features in a dataset while retaining variation across observations to preserve the information in that variation.

Solution:

C is correct. A is incorrect because it describes classification, not dimension reduction. B is incorrect because it describes clustering, not dimension reduction.

EVALUATING ML ALGORITHM PERFORMANCE

3

☐ describe overfitting and identify methods of addressing it

Machine learning algorithms promise several advantages relative to a structured statistical approach in exploring and analyzing the structure of very large datasets. ML algorithms have the ability to uncover complex interactions between feature variables and the target variable, and they can process massive amounts of data quickly. Moreover, many ML algorithms can easily capture non-linear relationships and may be able to recognize and predict structural changes between features and the target. These advantages mainly derive from the non-parametric and non-linear models that allow more flexibility when inferring relationships.

The flexibility of ML algorithms comes with a price, however. ML algorithms can produce overly complex models with results that are difficult to interpret, may be sensitive to noise or particulars of the data, and may fit the training data too well. An ML algorithm that fits the training data too well will typically not predict well using new data. This problem is known as **overfitting**, and it means that the fitted algorithm does not **generalize** well to new data. A model that generalizes well is a model that retains its explanatory power when predicting using out-of-sample (i.e., new) data. An overfit model has incorporated the noise or random fluctuations in the training data into its learned relationship. The problem is that these aspects often do not apply to

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new data the algorithm receives and so will negatively impact the model's ability to generalize, therefore reducing its overall predictive value. The evaluation of any ML algorithm thus focuses on its prediction error on new data rather than on its goodness of fit on the data with which the algorithm was fitted (i.e., trained).

Generalization is an objective in model building, so the problem of overfitting is a challenge to attaining that objective. These two concepts are the focus of the discussion below.

Generalization and Overfitting

To properly describe generalization and overfitting of an ML model, it is important to note the partitioning of the dataset to which the model will be applied. The dataset is typically divided into three non-overlapping samples: (1) **training sample** used to train the model, (2) **validation sample** for validating and tuning the model, and (3) **test sample** for testing the model's ability to predict well on new data. The training and validation samples are often referred to as being "in-sample," and the test sample is commonly referred to as being "out-of-sample." We will return shortly to the topic of partitioning the dataset.

To be valid and useful, any supervised machine learning model must generalize well beyond the training data. The model should retain its explanatory power when tested out-of-sample. As mentioned, one common reason for failure to generalize is overfitting. Think of overfitting as tailoring a custom suit that fits only one person. Continuing the analogy, underfitting is similar to making a baggy suit that fits no one, whereas robust fitting, the desired result, is similar to fashioning a universal suit that fits all people of similar dimensions.

The concepts of underfitting, overfitting, and good (or robust) fitting are illustrated in Exhibit 3. Underfitting means the model does not capture the relationships in the data. The left graph shows four errors in this underfit model (three misclassified circles and one misclassified triangle). Overfitting means training a model to such a degree of specificity to the training data that the model begins to incorporate noise coming from quirks or spurious correlations; it mistakes randomness for patterns and relationships. The algorithm may have memorized the data, rather than learned from it, so it has perfect hindsight but no foresight. The main contributors to overfitting are thus high noise levels in the data and too much complexity in the model. The middle graph shows no errors in this overfit model. **Complexity** refers to the number of features, terms, or branches in the model and to whether the model is linear or non-linear (non-linear is more complex). As models become more complex, overfitting risk increases. A good fit/robust model fits the training (in-sample) data well and generalizes well to out-of-sample data, both within acceptable degrees of error. The right graph shows that the good fitting model has only one error, the misclassified circle.

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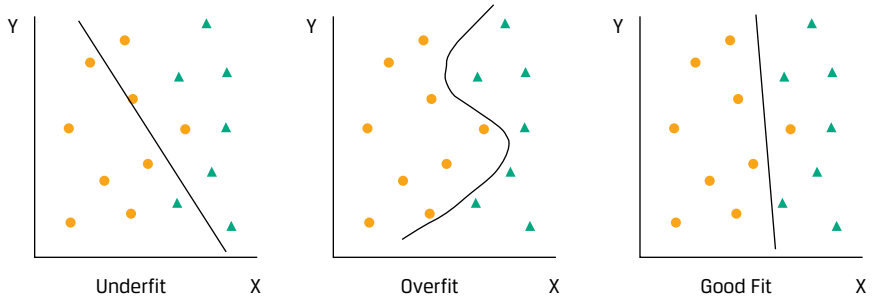
Exhibit 3: Underfitting, Overfitting, and Good Fitting



Errors and Overfitting

To capture these effects and calibrate degree of fit, data scientists compare error rates in- and out-of-sample as a function of both the data and the algorithm. Total in-sample errors (E_{in}) are generated by the predictions of the fitted relationship relative to actual target outcomes on the training sample. Total out-of-sample errors (E_{out}) are from either the validation or test samples. Low or no in-sample error but large out-of-sample error are indicative of poor generalization. Data scientists decompose the total out-of-sample error into three sources:

1. **Bias error**, or the degree to which a model fits the training data. Algorithms with erroneous assumptions produce high bias with poor approximation, causing underfitting and high in-sample error.
2. **Variance error**, or how much the model's results change in response to new data from validation and test samples. Unstable models pick up noise and produce high variance, causing overfitting and high out-of-sample error.
3. **Base error** due to randomness in the data.

A **learning curve** plots the accuracy rate (= 1 – error rate) in the validation or test samples (i.e., out-of-sample) against the amount of data in the training sample, so it is useful for describing under- and overfitting as a function of bias and variance errors. If the model is robust, out-of-sample accuracy increases as the training sample size increases. This implies that error rates experienced in the validation or test samples (E_{out}) and in the training sample (E_{in}) converge toward each other and toward a desired error rate (or, alternatively, the base error). In an underfitted model with high bias error, shown in the left panel of Exhibit 4, high error rates cause convergence below the desired accuracy rate. Adding more training samples will not improve the model to the desired performance level. In an overfitted model with high variance error, shown in the middle panel of Exhibit 4, the validation sample and training sample error rates fail to converge. In building models, data scientists try to simultaneously minimize both bias and variance errors while selecting an algorithm with good predictive or classifying power, as seen in the right panel of Exhibit 4.

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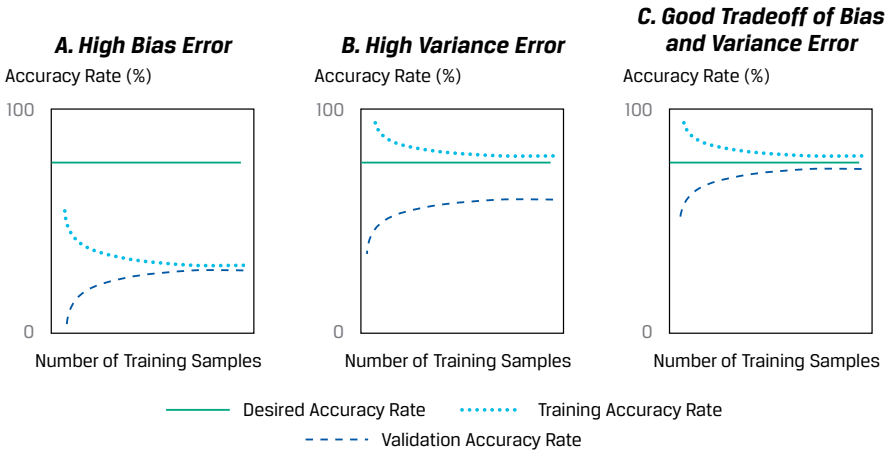
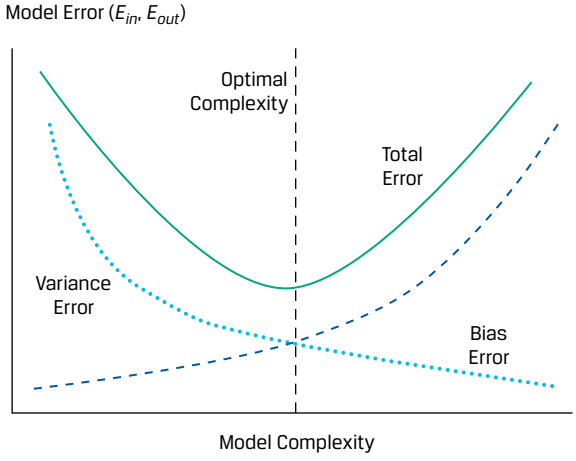
Exhibit 4: Learning Curves: Accuracy in Validation and Training Samples



Out-of-sample error rates are also a function of model complexity. As complexity increases in the training set, error rates (E_{in}) fall and bias error shrinks. As complexity increases in the test set, however, error rates (E_{out}) rise and variance error rises. Typically, linear functions are more susceptible to bias error and underfitting, while non-linear functions are more prone to variance error and overfitting. Therefore, an optimal point of model complexity exists where the bias and variance error curves intersect and in- and out-of-sample error rates are minimized. A **fitting curve**, which shows in- and out-of-sample error rates (E_{in} and E_{out}) on the y-axis plotted against model complexity on the x-axis, is presented in Exhibit 5 and illustrates this trade-off.

Exhibit 5: Fitting Curve Shows Trade-Off between Bias and Variance Errors and Model Complexity



Finding the optimal point (managing overfitting risk)—the point just before the total error rate starts to rise (due to increasing variance error)—is a core part of the machine learning process and the key to successful generalization. Data scientists express the trade-off between overfitting and generalization as a trade-off between

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cost (the difference between in- and out-of-sample error rates) and *complexity*. They use the trade-off between cost and complexity to calibrate and visualize under- and overfitting and to optimize their models.

Preventing Overfitting in Supervised Machine Learning

We have seen that overfitting impairs generalization, but overfitting potential is endemic to the supervised machine learning process due to the presence of noise. So, how do data scientists combat this risk? Two common methods are used to reduce overfitting: (1) preventing the algorithm from getting too complex during selection and training, which requires estimating an overfitting penalty, and (2) proper data sampling achieved by using **cross-validation**, a technique for estimating out-of-sample error directly by determining the error in validation samples.

The first strategy comes from Occam's razor, the problem-solving principle that the simplest solution tends to be the correct one. In supervised machine learning, it means limiting the number of features and penalizing algorithms that are too complex or too flexible by constraining them to include only parameters that reduce out-of-sample error.

The second strategy comes from the principle of avoiding sampling bias. But sampling bias can creep into machine learning in many ways. The challenge is having a large enough dataset to make both training and testing possible on representative samples. An unrepresentative sample or reducing the training sample size too much could obscure its true patterns, thereby increasing bias. In supervised machine learning, the technique for reducing sampling bias is through careful partitioning of the dataset into three groups: (1) training sample, the set of labeled training data where the target variable (Y) is known;(2) validation sample, the set of data used for making structural choices on the degree of model complexity, comparing various solutions, and tuning the selected model, thereby validating the model; and (3) test sample, the set of data held aside for testing to confirm the model's predictive or classifying power. The goal, of course, is to deploy the tested model on fresh data from the same domain.

To mitigate the problem of such **holdout samples** (i.e., data samples not used to train the model) reducing the training set size too much, modelers use special cross-validation techniques. One such technique is ***k*-fold cross-validation**, in which the data (excluding test sample and fresh data) are shuffled randomly and then are divided into k equal sub-samples, with $k - 1$ samples used as training samples and one sample, the kth, used as a validation sample. Note that k is typically set at 5 or 10. This process is then repeated k times, which helps minimize both bias and variance by insuring that each data point is used in the training set $k - 1$ times and in the validation set once. The average of the k validation errors (mean E_{val}) is then taken as a reasonable estimate of the model's out-of-sample error (E_{out}). A limitation of k-fold cross-validation is that it cannot be used with time-series data, where only the most recent data can reasonably be used for model validation.

In sum, mitigating overfitting risk by avoiding excessive out-of-sample error is critical to creating a supervised machine learning model that generalizes well to fresh datasets drawn from the same distribution. The main techniques used to mitigate overfitting risk in model construction are complexity reduction (or regularization) and cross-validation.

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EXAMPLE 2

Evaluating ML Algorithm Performance

Shreya Anand is a portfolio manager based in the Mumbai headquarters office of an investment firm, where she runs a high-dividend-yield fund for wealthy clients. Anand has some knowledge of data science from her university studies. She is interested in classifying companies in the NIFTY 200 Index—an index of large- and mid-cap companies listed on the National Stock Exchange of India—into two categories: dividend increase and no dividend increase. She assembles data for training, validating, and testing an ML-based model that consists of 1,000 observations of NIFTY 200 companies, each consisting of 25 features (fundamental and technical) and the labeled target (dividend increase or no dividend increase).

After training her model, Anand discovers that while it is good at correctly classifying using the training sample, it does not perform well on new data. In consulting her colleagues about this issue, Anand hears conflicting explanations about what constitutes good generalization in an ML model:

Statement 1	The model retains its explanatory power when predicting using new data (i.e., out-of-sample).
Statement 2	The model shows low explanatory power after training using in-sample data (i.e., training data).
Statement 3	The model loses its explanatory power when predicting using new data (i.e., out-of-sample).

1. Which statement made to Anand is *most* accurate?

   A. Statement 1

   B. Statement 2

   C. Statement 3

Solution:

A, Statement 1, is correct. B, Statement 2, is incorrect because it describes a poorly fitting model with high bias. C, Statement 3, is incorrect because it describes an overfitted model with poor generalization.

2. Anand's model is *most likely* being impaired by which of the following?

   A. Underfitting and bias error

   B. Overfitting and variance error

   C. Overfitting and bias error

Solution:

B is correct. Anand's model is good at correctly classifying using the training sample, but it does not perform well using new data. The model is overfitted, so it has high variance error.

3. By implementing which one of the following actions can Anand address the problem?

   A. Estimate and incorporate into the model a penalty that decreases in size with the number of included features.

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B. Use the *k*-fold cross-validation technique to estimate the model's out-of-sample error, and then adjust the model accordingly.

C. Use an unsupervised learning model.

Solution:

B is correct. A is incorrect because the penalty should increase in size with the number of included features. C is incorrect because Anand is using labeled data for classification, and unsupervised learning models do not use labeled data.

SUPERVISED ML ALGORITHMS: PENALIZED REGRESSION

4

☐ describe supervised machine learning algorithms—including penalized regression, support vector machine, k-nearest neighbor, classification and regression tree, ensemble learning, and random forest—and determine the problems for which they are best suited

Supervised machine learning models are trained using labeled data, and depending on the nature of the target (*Y*) variable, they can be divided into two types: regression for a continuous target variable and classification for a categorical or ordinal target variable. As shown in Exhibit 2 under regression, we will now cover penalized regression and LASSO. Then, as shown under classification, we will introduce support vector machine (SVM), *k*-nearest neighbor (KNN), and classification and regression tree (CART) algorithms. Note that CART, as its name implies, can be used for both classification and regression problems.

In the following discussion, assume we have a number of observations of a target variable, *Y*, and *n* real valued features, $X_1, \ldots, X_n$, that we may use to establish a relationship (regression or classification) between X (a vector of the X_i) and *Y* for each observation in our dataset.

Penalized Regression

Penalized regression is a computationally efficient technique used in prediction problems. In practice, penalized regression has been useful for reducing a large number of features to a manageable set and for making good predictions in a variety of large datasets, especially where features are correlated (i.e., when classical linear regression breaks down).

In a large dataset context, we may have many features that potentially could be used to explain *Y*. When a model is fit to training data, the model may so closely reflect the characteristics of the specific training data that the model does not perform well on new data. Features may be included that reflect noise or randomness in the training dataset that will not be present in new or future data used for making predictions. That is the problem of overfitting, and penalized regression can be described as a technique to avoid overfitting. In prediction, out-of-sample performance is key, so relatively parsimonious models (that is, models in which each variable plays an essential role) tend to work well because they are less subject to overfitting.

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Let us suppose that we standardize our data so the features have a mean of 0 and a variance of 1. Standardization of features will allow us to compare the magnitudes of regression coefficients for the feature variables. In ordinary linear regression (i.e., ordinary least squares, or OLS), the regression coefficients $\hat{b}_0, \hat{b}_1, \ldots, \hat{b}_K$ are chosen to *minimize* the sum of the squared residuals (i.e., the sum of the squared difference between the actual values, Y_i, and the predicted values, $\hat{Y}_i$), or

$$\sum_{i=1}^{n} \left(Y_i - \hat{Y}_i\right)^2.$$

Penalized regression includes a constraint such that the regression coefficients are chosen to minimize the sum of squared residuals *plus* a penalty term that increases in size with the number of included features. So, in a penalized regression, a feature must make a sufficient contribution to model fit to offset the penalty from including it. Therefore, only the more important features for explaining Y will remain in the penalized regression model.

In one popular type of penalized regression, **LASSO**, or least absolute shrinkage and selection operator, the penalty term has the following form, with $\lambda > 0$:

$$\text{Penalty term} = \lambda \sum_{k=1}^{K} \left|\hat{b}_k\right|.$$

In addition to minimizing the sum of the squared residuals, LASSO involves minimizing the sum of the absolute values of the regression coefficients (see the following expression). The greater the number of included features (i.e., variables with non-zero coefficients), the larger the penalty term. Therefore, penalized regression ensures that a feature is included only if the sum of squared residuals declines by more than the penalty term increases. All types of penalized regression involve a trade-off of this type. Also, since LASSO eliminates the least important features from the model, it automatically performs a type of feature selection.

$$\sum_{i=1}^{n} \left(Y_i - \hat{Y}_i\right)^2 + \lambda \sum_{k=1}^{K} \left|\hat{b}_k\right|.$$

Lambda (λ) is a **hyperparameter**—a parameter whose value must be set by the researcher before learning begins—of the regression model and will determine the balance between fitting the model versus keeping the model parsimonious. In practice, a hyperparameter is set by reviewing model performance repeatedly at different settings on the validation set, and hence the test set is also essential to avoid overfitting of hyperparameters to the validation data.

Note that in the case where $\lambda = 0$, the LASSO penalized regression is equivalent to an OLS regression. When using LASSO or other penalized regression techniques, the penalty term is added only during the model building process (i.e., when fitting the model to the training data). Once the model has been built, the penalty term is no longer needed, and the model is then evaluated by the sum of the squared residuals generated using the test dataset.

With today's availability of fast computation algorithms, investment analysts are increasingly using LASSO and other regularization techniques to remove less pertinent features and build parsimonious models. **Regularization** describes methods that reduce statistical variability in high-dimensional data estimation problems—in this case, reducing regression coefficient estimates toward zero and thereby avoiding complex models and the risk of overfitting. LASSO has been used, for example, for forecasting default probabilities in industrial sectors where scores of potential features, many collinear, have been reduced to fewer than 10 variables, which is important given the relatively small number (about 100) of observations of default.

Regularization methods can also be applied to non-linear models. A long-term challenge of the asset management industry in applying mean–variance optimization has been the estimation of stable covariance matrixes and asset weights for large

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portfolios. Asset returns typically exhibit strong multi-collinearity, making the estimation of the covariance matrix highly sensitive to noise and outliers, so the resulting optimized asset weights are highly unstable. Regularization methods have been used to address this problem. The relatively parsimonious models produced by applying penalized regression methods, such as LASSO, tend to work well because they are less subject to overfitting.

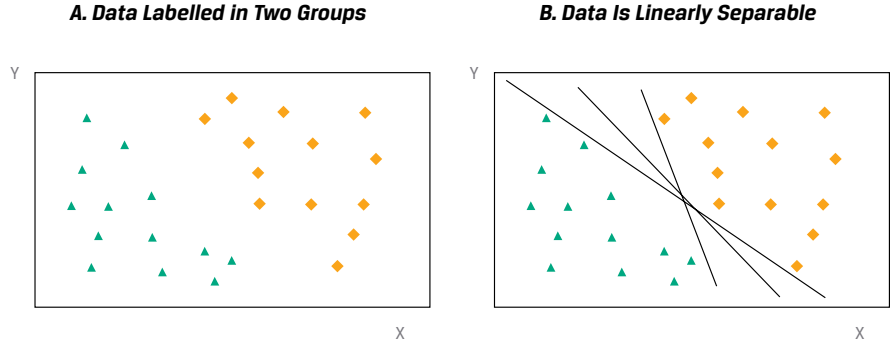
SUPPORT VECTOR MACHINE

5

☐ describe supervised machine learning algorithms—including penalized regression, support vector machine, k-nearest neighbor, classification and regression tree, ensemble learning, and random forest—and determine the problems for which they are best suited

Support vector machine (SVM) is one of the most popular algorithms in machine learning. It is a powerful supervised algorithm used for classification, regression, and outlier detection. Despite its complicated-sounding name, the notion is relatively straightforward and best explained with a few pictures. The left panel in Exhibit 6 presents a simple dataset with two features (x and y coordinates) labeled in two groups (triangles and diamonds). These binary labeled data are noticeably separated into two distinct regions, which could represent stocks with positive and negative returns in a given year. These two regions can be easily separated by an infinite number of straight lines; three of them are shown in the right panel of Exhibit 6. The data are thus linearly separable, and any of the straight lines shown would be called a **linear classifier**—a binary classifier that makes its classification decision based on a linear combination of the features of each data point.

Exhibit 6: Scatterplots and Linear Separation of Labeled Data



With two dimensions or features (x and y), linear classifiers can be represented as straight lines. Observations with n features can be represented in an n-dimension space, and the dataset would be linearly separable if the observations can be separated into two distinct regions by a linear space boundary. The general term for such a space boundary is an n-dimensional hyperplane, which with $n = 1$ is called a line and with $n = 2$ is called a plane.

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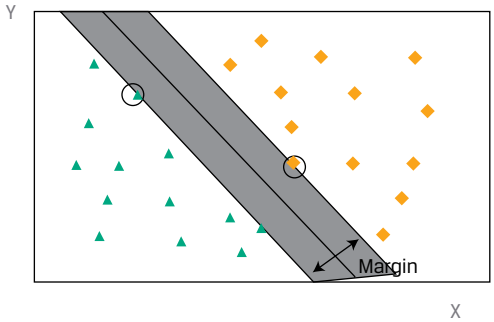
Support vector machine is a linear classifier that determines the hyperplane that optimally separates the observations into two sets of data points. The intuitive idea behind the SVM algorithm is maximizing the probability of making a correct prediction (here, that an observation is a triangle or a diamond) by determining the boundary that is the furthest away from all the observations. In Exhibit 7, SVM separates the data by the maximum margin, where the margin is the shaded strip that divides the observations into two groups. The straight line in the middle of the shaded strip is the discriminant boundary, or boundary, for short. We can see that the SVM algorithm produces the widest shaded strip (i.e., the one with the maximum margin on either side of the boundary). The margin is determined by the observations closest to the boundary (the circled points) in each set, and these observations are called support vectors. Adding more training data away from the support vectors will not affect the boundary. In our training datasets, however, adding data points which are close to the hyperplane may move the margin by changing the set of support vectors.

Exhibit 7: Linear Support Vector Machine Classifier



In Exhibit 7, SVM is classifying all observations perfectly. Most real-world datasets, however, are not linearly separable. Some observations may fall on the wrong side of the boundary and be misclassified by the SVM algorithm. The SVM algorithm handles this problem by an adaptation called **soft margin classification**, which adds a penalty to the objective function for observations in the training set that are misclassified. In essence, the SVM algorithm will choose a discriminant boundary that optimizes the trade-off between a wider margin and a lower total error penalty.

As an alternative to soft margin classification, a non-linear SVM algorithm can be run by introducing more advanced, non-linear separation boundaries. These algorithms may reduce the number of misclassified instances in the training datasets but are more complex and, so, are prone to overfitting.

SVM has many applications in investment management. It is particularly suited for small to medium-size but complex high-dimensional datasets, such as corporate financial statements or bankruptcy databases. Investors seek to predict company failures for identifying stocks to avoid or to short sell, and SVM can generate a binary classification (e.g., bankruptcy likely versus bankruptcy unlikely) using many fundamental and technical feature variables. SVM can effectively capture the characteristics of such data with many features while being resilient to outliers and correlated features. SVM can also be used to classify text from documents (e.g., news articles, company announcements, and company annual reports) into useful categories for investors (e.g., positive sentiment and negative sentiment).

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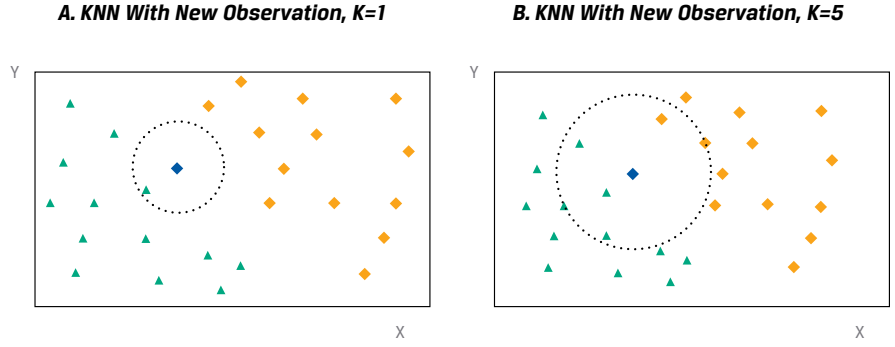
K-NEAREST NEIGHBOR

6

- describe supervised machine learning algorithms—including penalized regression, support vector machine, k-nearest neighbor, classification and regression tree, ensemble learning, and random forest—and determine the problems for which they are best suited

***K*-nearest neighbor** (KNN) is a supervised learning technique used most often for classification and sometimes for regression. The idea is to classify a new observation by finding similarities ("nearness") between this new observation and the existing data. Going back to the scatterplot in Exhibit 6, let us assume we have a new observation: The diamond in Exhibit 8 needs to be classified as belonging to either the diamond or the triangle category. If $k = 1$, the diamond will be classified into the same category as its nearest neighbor (i.e., the triangle in the left panel). The right panel in Exhibit 8 presents the case where $k = 5$, so the algorithm will look at the diamond's five nearest neighbors, which are three triangles and two diamonds. The decision rule is to choose the classification with the largest number of nearest neighbors out of the five being considered. So, the diamond is again classified as belonging to the triangle category.

Exhibit 8: *K*-Nearest Neighbor Algorithm



Let us suppose we have a database of corporate bonds classified by credit rating that also contains detailed information on the characteristics of these bonds. Such features would include those of the issuing company (e.g., asset size, industry, leverage ratios, cash flow ratios) and of the bond issue itself (e.g., tenor, fixed/floating coupon, embedded options). Now, assume a new bond is about to be issued with no credit rating. By nature, corporate bonds with similar issuer and issue characteristics should be given a similar credit rating. So, by using KNN, we can predict the implied credit rating of the new bond based on the similarities of its characteristics to those of the bonds in our database.

KNN is a straightforward, intuitive model that is still very powerful because it is non-parametric; the model makes no assumptions about the distribution of the data. Moreover, it can be used directly for multi-class classification. A critical challenge of KNN, however, is defining what it means to be "similar" (or near). Besides the selection of features, an important decision relates to the distance metric used to model similarity because an inappropriate measure will generate poorly performing models. The choice of a correct distance measure may be even more subjective for ordinal

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or categorical data. For example, if an analyst is looking at the similarities in market performance of various equities, he or she may consider using the correlation between the stocks' historical returns as an appropriate measure of similarity.

Knowledge of the data and understanding of the business objectives of the analysis are critical aspects in the process of defining similarity. KNN results can be sensitive to inclusion of irrelevant or correlated features, so it may be necessary to select features manually. By doing so, the analyst removes less valuable information to keep the most relevant and pertinent information. If done correctly, this process should generate a more representative distance measure. KNN algorithms tend to work better with a small number of features.

Finally, the number *k*, the hyperparameter of the model, must be chosen with the understanding that different values of *k* can lead to different conclusions. For predicting the credit rating of an unrated bond, for example, should *k* be the 3, 15, or 50 bonds most similar to the unrated bond? If *k* is an even number, there may be ties and no clear classification. Choosing a value for *k* that is too small would result in a high error rate and sensitivity to local outliers, but choosing a value for *k* that is too large would dilute the concept of nearest neighbors by averaging too many outcomes. In practice, several different techniques can be used to determine an optimal value for *k*, taking into account the number of categories and their partitioning of the feature space.

The KNN algorithm has many applications in the investment industry, including bankruptcy prediction, stock price prediction, corporate bond credit rating assignment, and customized equity and bond index creation. For example, KNN is useful for determining bonds that are similar and those that are dissimilar, which is critical information for creating a custom, diversified bond index.

7 CLASSIFICATION AND REGRESSION TREE

☐ describe supervised machine learning algorithms—including penalized regression, support vector machine, k-nearest neighbor, classification and regression tree, ensemble learning, and random forest—and determine the problems for which they are best suited

Classification and regression tree (CART) is another common supervised machine learning technique that can be applied to predict either a categorical target variable, producing a classification tree, or a continuous target variable, producing a regression tree. CART is commonly applied to binary classification or regression.

CART will be discussed in the context of a simplified model for classifying companies by whether they are likely to increase their dividends to shareholders. Such a classification requires a binary tree: a combination of an initial root node, decision nodes, and terminal nodes. The root node and each decision node represent a single feature (*f*) and a cutoff value (*c*) for that feature. As shown in Panel A of Exhibit 9, we start at the initial root node for a new data point. In this case, the initial root node represents the feature investment opportunities growth (IOG), designated as X1, with a cutoff value of 10%. From the initial root node, the data are partitioned at decision nodes into smaller and smaller subgroups until terminal nodes that contain the predicted labels are formed. In this case, the predicted labels are either dividend increase (the cross) or no dividend increase (the dash).

Also shown in Panel A of Exhibit 9, if the value of feature IOG (X1) is greater than 10% (Yes), then we proceed to the decision node for free cash flow growth (FCFG), designated as X2, which has a cutoff value of 20%. Now, if the value of FCFG is not

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greater than 20% (No), then CART will predict that that data point belongs to the no dividend increase (dash) category, which represents a terminal node. Conversely, if the value of X2 is greater than 20% (Yes), then CART will predict that that data point belongs to the dividend increase (cross) category, which represents another terminal node.

It is important to note that the same feature can appear several times in a tree in combination with other features. Moreover, some features may be relevant only if other conditions have been met. For example, going back to the initial root node, if IOG is not greater than 10% (X1 ≤ 10%) and FCFG is greater than 10%, then IOG appears again as another decision node, but this time it is lower down in the tree and has a cutoff value of 5%.

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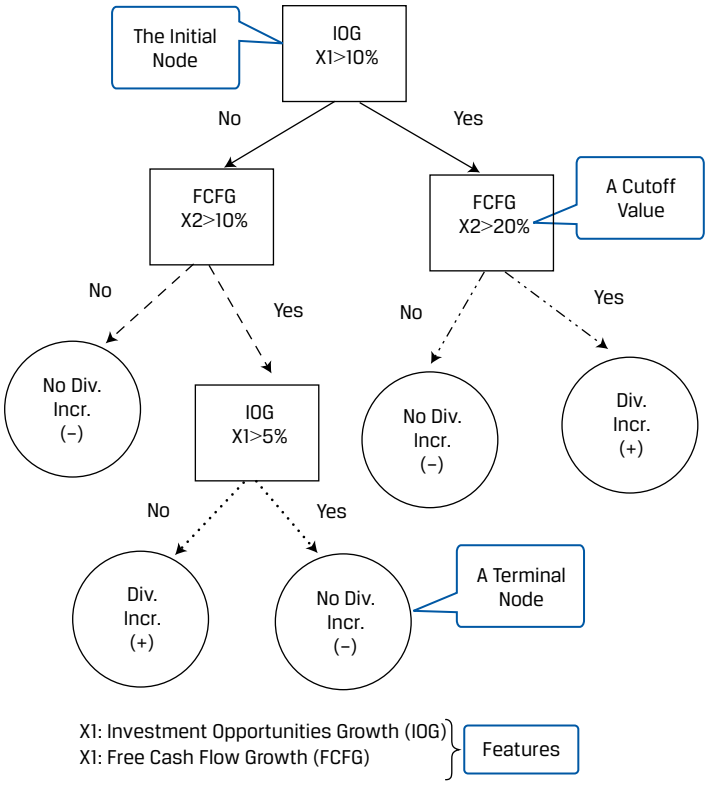
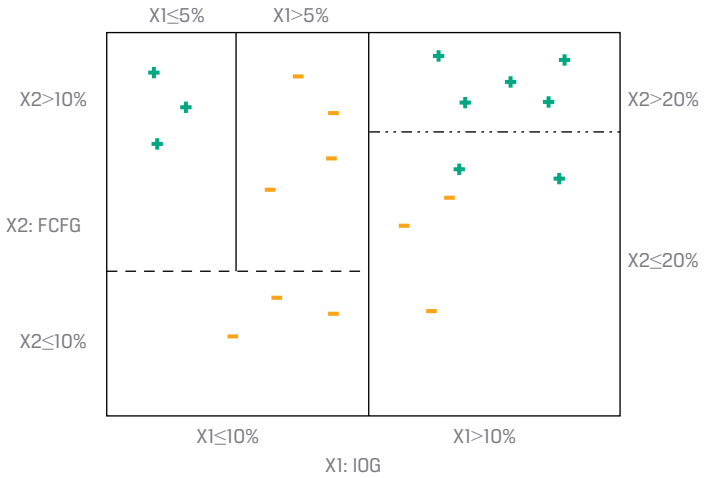
Exhibit 9: Classification and Regression Tree—Decision Tree and Partitioning of the Feature Space

A. Decision Tree



B. Partitioning of the Feature (X1, X2) Space



We now turn to how the CART algorithm selects features and cutoff values for them. Initially, the classification model is trained from the labeled data, which in this hypothetical case are 10 instances of companies having a dividend increase (the crosses) and 10 instances of companies with no dividend increase (the dashes). As shown in Panel B of Exhibit 9, at the initial root node and at each decision node, the feature space (i.e., the plane defined by X1 and X2) is split into two rectangles for values above and below the cutoff value for the particular feature represented at that node.

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This can be seen by noting the distinct patterns of the lines that emanate from the decision nodes in Panel A. These same distinct patterns are used for partitioning the feature space in Panel B.

The CART algorithm chooses the feature and the cutoff value at each node that generates the widest separation of the labeled data to minimize classification error (e.g., by a criterion, such as mean-squared error). After each decision node, the partition of the feature space becomes smaller and smaller, so observations in each group have lower within-group error than before. At any level of the tree, when the classification error does not diminish much more from another split (bifurcation), the process stops, the node is a terminal node, and the category that is in the majority at that node is assigned to it. If the objective of the model is classification, then the prediction of the algorithm at each terminal node will be the category with the majority of data points. For example, in Panel B of Exhibit 9, the top right rectangle of the feature space, representing IOG (X1) > 10% and FCFG (X2)> 20%, contains five crosses, the most data points of any of the partitions. So, CART would predict that a new data point (i.e., a company) with such features belongs to the dividend increase (cross) category. However, if instead the new data point had IOG (X1) > 10% and FCFG (X2) ≤ 20%, then it would be predicted to belong to the no dividend increase (dash) category—represented by the lower right rectangle, with two crosses but with three dashes. Finally, if the goal is regression, then the prediction at each terminal node is the mean of the labeled values.

CART makes no assumptions about the characteristics of the training data, so if left unconstrained, it potentially can perfectly learn the training data. To avoid such overfitting, regularization parameters can be added, such as the maximum depth of the tree, the minimum population at a node, or the maximum number of decision nodes. The iterative process of building the tree is stopped once the regularization criterion has been reached. For example, in Panel B of Exhibit 9, the upper left rectangle of the feature space (determined by X1 ≤ 10%, X2 > 10%, and X1 ≤ 5% with three crosses) might represent a terminal node resulting from a regularization criterion with minimum population equal to 3. Alternatively, regularization can occur via a **pruning** technique that can be used afterward to reduce the size of the tree. Sections of the tree that provide little classifying power are pruned (i.e., cut back or removed).

By its iterative structure, CART can uncover complex dependencies between features that other models cannot reveal. As demonstrated in Exhibit 9, the same feature can appear several times in combination with other features and some features may be relevant only if other conditions have been met.

As shown in Exhibit 10, high profitability is a critical feature for predicting whether a stock is an attractive investment or a value trap (i.e., an investment that, although apparently priced cheaply, is likely to be unprofitable). This feature is relevant only if the stock is cheap: For example, in this hypothetical case, if P/E is less than 15, leverage is high (debt to total capital > 50%) and sales are expanding (sales growth > 15%). Said another way, high profitability is irrelevant in this context if the stock is not cheap *and* if leverage is not high *and* if sales are not expanding. Multiple linear regression typically fails in such situations where the relationship between the features and the outcome is non-linear.

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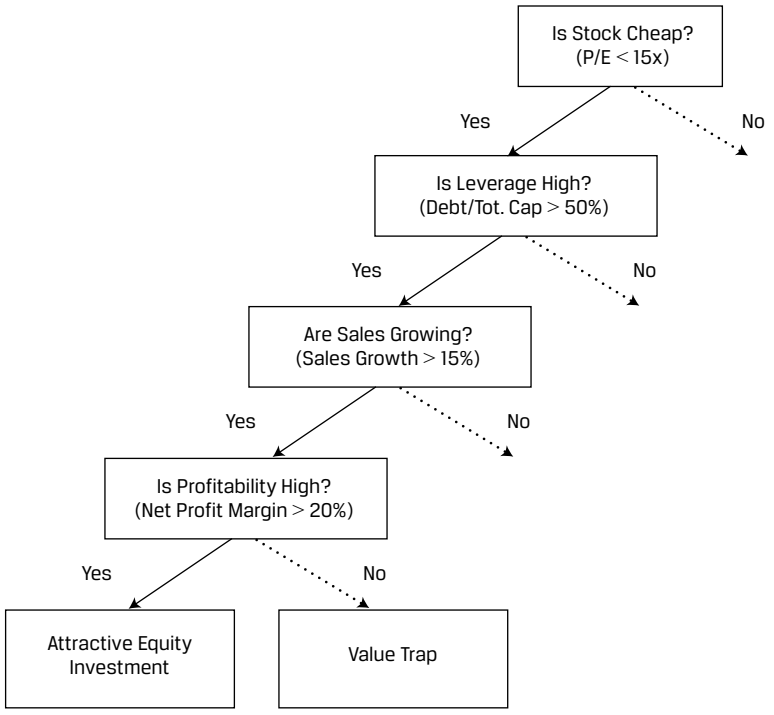
Exhibit 10: Stylized Decision Tree—Attractive Investment or Value Trap?



CART models are popular supervised machine learning models because the tree provides a visual explanation for the prediction. This contrasts favorably with other algorithms that are often considered to be "black boxes" because it may be difficult to understand the reasoning behind their outcomes and thus to place trust in them. CART is a powerful tool to build expert systems for decision-making processes. It can induce robust rules despite noisy data and complex relationships between high numbers of features. Typical applications of CART in investment management include, among others, enhancing detection of fraud in financial statements, generating consistent decision processes in equity and fixed-income selection, and simplifying communication of investment strategies to clients.

8 ENSEMBLE LEARNING AND RANDOM FOREST

☐ describe supervised machine learning algorithms—including penalized regression, support vector machine, k-nearest neighbor, classification and regression tree, ensemble learning, and random forest—and determine the problems for which they are best suited

Instead of basing predictions on the results of a single model as in the previous discussion, why not use the predictions of a group—or an ensemble—of models? Each single model will have a certain error rate and will make noisy predictions. But by taking the average result of many predictions from many models, we can expect to achieve a reduction in noise as the average result converges toward a more accurate prediction. This technique of combining the predictions from a collection of models

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is called **ensemble learning**, and the combination of multiple learning algorithms is known as the **ensemble method**. Ensemble learning typically produces more accurate and more stable predictions than the best single model. In fact, in many prestigious machine learning competitions, an ensemble method is often the winning solution.

Ensemble learning can be divided into two main categories: (1) aggregation of heterogeneous learners (i.e., different types of algorithms combined with a voting classifier) or (2) aggregation of homogeneous learners (i.e., a combination of the same algorithm using different training data that are based, for example, on a bootstrap aggregating, or bagging, technique, as discussed later).

Voting Classifiers

Suppose you have been working on a machine learning project for some time and have trained and compared the results of several algorithms, such as SVM, KNN, and CART. A **majority-vote classifier** will assign to a new data point the predicted label with the most votes. For example, if the SVM and KNN models are both predicting the category "stock outperformance" and the CART model is predicting the category "stock underperformance," then the majority-vote classifier will choose stock outperformance." The more individual models you have trained, the higher the accuracy of the aggregated prediction up to a point. There is an optimal number of models beyond which performance would be expected to deteriorate from overfitting. The trick is to look for diversity in the choice of algorithms, modeling techniques, and hypotheses. The (extreme) assumption here is that if the predictions of the individual models are independent, then we can use the law of large numbers to achieve a more accurate prediction.

Bootstrap Aggregating (Bagging)

Alternatively, one can use the same machine learning algorithm but with different training data. **Bootstrap aggregating (or bagging)** is a technique whereby the original training dataset is used to generate n new training datasets or bags of data. Each new bag of data is generated by random sampling with replacement from the initial training set. The algorithm can now be trained on n independent datasets that will generate n new models. Then, for each new observation, we can aggregate the n predictions using a majority-vote classifier for a classification or an average for a regression. Bagging is a very useful technique because it helps to improve the stability of predictions and protects against overfitting the model.

Random Forest

A **random forest classifier** is a collection of a large number of decision trees trained via a bagging method. For example, a CART algorithm would be trained using each of the n independent datasets (from the bagging process) to generate the multitude of different decision trees that make up the random forest classifier.

To derive even more individual predictions, added diversity can be generated in the trees by randomly reducing the number of features available during training. So, if each observation has n features, one can randomly select a subset of m features (where $m < n$) that will then be considered by the CART algorithm for splitting the dataset at each of the decision nodes. The number of subset features (m), the number of trees to use, the minimum size (population) of each node (or leaf), and the maximum depth of each tree are all hyperparameters that can be tuned to improve overall model prediction accuracy. For any new observation, we let all the classifier trees (the "random forest") undertake classification by majority vote—implementing a machine learning version of the "wisdom of crowds." The process involved in random forest

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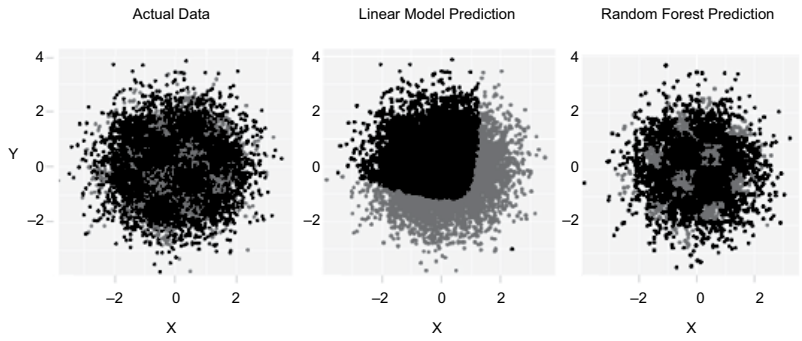
construction tends to reduce variance and protect against overfitting on the training data. It also reduces the ratio of noise to signal because errors cancel out across the collection of slightly different classification trees. However, an important drawback of random forest is that it lacks the ease of interpretability of individual trees; as a result, it is considered a relatively black box type of algorithm.

Exhibit 11 presents three scatterplots of actual and predicted defaults by small and medium-sized businesses with respect to two features, X and Y—for example, firm profitability and leverage, respectively. The left plot shows the actual cases of default in light shade and no default in dark shade, while the middle and right plots present the predicted defaults and no defaults (also in light and dark shades, respectively). It is clear from the middle plot, which is based on a traditional linear regression model, that the model fails to predict the complex non-linear relationship between the features. Conversely, the right plot, which presents the prediction results of a random forest model, shows that this model performs very well in matching the actual distribution of the data.

Exhibit 11: Credit Defaults of Small- and Medium-Sized Borrowers



Source: Bacham and Zhao (2017).

ENSEMBLE LEARNING WITH RANDOM FOREST

In making use of voting across classifier trees, random forest is an example of ensemble learning: Incorporating the output of a collection of models produces classifications that have better signal-to-noise ratios than the individual classifiers. A good example is a credit card fraud detection problem that comes from an open source dataset on Kaggle.[1] Here, the data contained several anonymized features that might be used to explain which transactions were fraudulent. The difficulty in the analysis arises from the fact that the rate of fraudulent transactions is very low; in a sample of 284,807 transactions, only 492 were fraudulent (0.17%). This is akin to finding a needle in a haystack. Applying a random forest classification algorithm with an oversampling technique—which involves increasing the proportional representation of fraudulent data in the training set—does extremely well. Despite the lopsided sample, it delivers **precision** (the

1 See www.kaggle.com/mlg-ulb/creditcardfraud (accessed 1 October 2018).

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ratio of correctly predicted fraudulent cases to all predicted fraudulent cases) of 89% and **recall** (the ratio of correctly predicted fraudulent cases to all actual fraudulent cases) of 82%.

Despite its relative simplicity, random forest is a powerful algorithm with many investment applications. These include, for example, use in factor-based investment strategies for asset allocation and investment selection or use in predicting whether an IPO will be successful (e.g., percent oversubscribed, first trading day close/IPO price) given the attributes of the IPO offering and the corporate issuer. Later, in a mini-case study, Deep Neural Network–Based Equity Factor Model, we present further details of how supervised machine learning is used for fundamental factor modeling.

EXAMPLE 3

Support Vector Machine and *K*-Nearest Neighbor

Rachel Lee is a fixed-income portfolio manager with Zeta Investment Management Company. Zeta manages an investment-grade bond portfolio for small, conservative institutions and a non-investment-grade (i.e., high-yield) bond portfolio for yield-seeking, high-net-worth individuals. Both portfolios can hold unrated bonds if the characteristics of the unrated bonds closely match those of the respective portfolio's average holding.

Lee is discussing an upcoming straight, 10-year fixed-coupon bond issue with senior credit analyst Marc Watson. Watson comments that although the bond's issuer, Biotron Corporation, has not had this issue rated, his analysis of the company's profitability, cash flow, leverage, and coverage ratios places the issue near the borderline between low investment-grade (Baa3/BBB–) and high non-investment-grade (Ba1/BB+) bonds.

Lee decides to use machine learning methods to confirm the implied credit rating of Biotron Corporation.

Lee decides to apply the two identified ML algorithms. Both algorithms clearly support a high non-investment-grade rating. Watson states that because both ML algorithms agree on the rating, he has confidence in relying on the rating.

1. State the type of problem being addressed by Lee.

Solution:

Lee is addressing a supervised learning classification problem because she must determine whether Biotron's upcoming bond issue would be classified as investment grade or non-investment grade.

2. State two ML algorithms that Lee could use to explore the implied credit rating of Biotron Corporation, and then describe how each algorithm could be applied.

Solution:

One suitable ML algorithm is SVM. The SVM algorithm is a linear classifier that aims to find the optimal hyperplane—the one that separates observations into two distinct sets by the maximum margin. So, SVM is well suited to binary classification problems, such as the one facing Lee (investment grade versus non-investment grade). In this case, Lee could train the SVM algorithm on data—characteristics (features) and rating (target)—of low investment-grade (Baa3/BBB–) and high non-investment-grade (Ba1/BB+)

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bonds. Lee would then note on which side of the margin the new data point (Biotron's new bonds) lies.

The KNN algorithm is also well suited for classification problems because it classifies a new observation by finding similarities (or nearness) between the new observation and the existing data. Training the algorithm with data as for SVM, the decision rule for classifying Biotron's new bonds is which classification is in the majority among its *k*-nearest neighbors. Note that *k* (a hyperparameter) must be pre-specified by Lee.

3. State one argument in support of Watson's viewpoint.

Solution:

If the ML algorithms disagreed on the classification, the classification would be more likely to be sensitive to the algorithm's approach to classifying data. Because the classification of Biotron's new issue appears robust to the choice of ML algorithm (i.e., both algorithms agree on the rating), the resulting classification will more likely be correct.

EXAMPLE 4

CART and Ensemble Learning

Laurie Kim is a portfolio manager at Hilux LLC, a high-yield bond investment firm. The economy has been in recession for several months, and high-yield bond prices have declined precipitously as credit spreads have widened in response to the weak macroeconomic environment. Kim, however, believes this is a good time to buy because she expects to profit as credit spreads narrow and high-yield bond prices rise in anticipation of economic recovery.

Based on her analysis, Kim believes that corporate high-yield bonds in the credit quality range of B/B2 to CCC/Caa2 are the most attractive. However, she must carefully select which bonds to buy and which bonds to avoid because of the elevated default risk caused by the currently weak economy.

To help with her bond selection, Kim turns to Hilux's data analytics team. Kim has supplied them with historical data consisting of 19 fundamental and 5 technical factors for several thousand high-yield bond issuers and issues labeled to indicate default or no default. Kim requests that the team develop an ML-based model using all the factors provided that will make accurate classifications in two categories: default and no default. Exploratory data analysis suggests considerable non-linearities among the feature set.

1. State the type of problem being addressed by Kim.

Solution:

Kim is addressing a classification problem because she must determine whether bonds that she is considering purchasing in the credit quality range of B/B2 to CCC/Caa2 will default or not default.

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2. Describe the dimensionality of the model that Kim requests her analytics team to develop.

Solution:

With 19 fundamental and 5 technical factors (i.e., the features), the dimensionality of the model is 24.

3. Evaluate whether a CART model is appropriate for addressing her problem.

Solution:

The CART model is an algorithm for addressing classification problems. Its ability to handle complex, non-linear relationships makes it a good choice to address the modeling problem at hand. An important advantage of CART is that its results are relatively straightforward to visualize and interpret, which should help Kim explain her recommendations based on the model to Hilux's investment committee and the firm's clients.

4. Describe how a CART model operates at each node of the tree.

Solution:

At each node in the decision tree, the algorithm will choose the feature and the cutoff value for the selected feature that generates the widest separation of the labeled data to minimize classification error.

5. Describe how the team might avoid overfitting and improve the predictive power of a CART model.

Solution:

The team can avoid overfitting and improve the predictive power of the CART model by adding regularization parameters. For example, the team could specify the maximum depth of the tree, the minimum population at a node, or the maximum number of decision nodes. The iterative process of building nodes will be stopped once the regularization criterion has been reached. Alternatively, a pruning technique can be used afterward to remove parts of the CART model that provide little power to correctly classify instances into default or no default categories.

6. Describe how ensemble learning might be used by the team to develop even better predictions for Kim's selection of corporate high-yield bonds.

Solution:

The analytics team might use ensemble learning to combine the predictions from a collection of models, where the average result of many predictions leads to a reduction in noise and thus more accurate predictions. Ensemble learning can be achieved by an aggregation of either heterogeneous learners—different types of algorithms combined with a voting classifier—or homogeneous learners—a combination of the same algorithm but using different training data based on the bootstrap aggregating (i.e., bagging) technique. The team may also consider developing a random forest classifier (i.e., a collection of many decision trees) trained via a bagging method.

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9 CASE STUDY: CLASSIFICATION OF WINNING AND LOSING FUNDS

☐ describe supervised machine learning algorithms—including penalized regression, support vector machine, k-nearest neighbor, classification and regression tree, ensemble learning, and random forest—and determine the problems for which they are best suited

The following case study was developed and written by Matthew Dixon, PhD, FRM.

A research analyst for a fund of funds has been tasked with identifying a set of attractive exchange-traded funds (ETFs) and mutual funds (MFs) in which to invest. She decides to use machine learning to identify the best (i.e., winners) and worst (i.e., losers) performing funds and the features which are most important in such an identification. Her aim is to train a model to correctly classify the winners and losers and then to use it to predict future outperformers. She is unsure of which type of machine learning classification model (i.e., classifier) would work best, so she reports and cross-compares her findings using several different well-known machine learning algorithms.

The goal of this case is to demonstrate the application of machine learning classification to fund selection. Therefore, the analyst will use the following classifiers to identify the best and worst performing funds:

- classification and regression tree (CART),
- support vector machine (SVM),
- *k*-nearest neighbors (KNN), and
- random forests.

Data Description

In the following experiments, the performance of each fund is learned by the machine learning algorithms based on fund type and size, asset class composition, fundamentals (i.e., valuation multiples), and sector composition characteristics. To form a cross-sectional classifier, the sector composition and fund size reported on 15 February 2019 are assumed to be representative of the latest month over which the fund return is reported. Exhibit 12 presents a description of the dataset.

Exhibit 12: Dataset Description

Dataset: MF and ETF Data

There are two separate datasets, one for MFs and one for ETFs, consisting of fund type, size, asset class composition, fundamental financial ratios, sector weights, and monthly total return labeled to indicate the fund as being a winner, a loser, or neither. Number of observations: 6,085 MFs and 1,594 ETFs.

Features: Up to 21, as shown below:

General (six features):

1. cat investment*: Fund type, either "blend," "growth," or "value"
2. net assets: Total net assets in US dollars

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3. cat size: Investment category size, either "small," "medium," or "large" market capitalization stocks
4. portfolio cash**: The ratio of cash to total assets in the fund
5. portfolio stocks: The ratio of stocks to total assets in the fund
6. portfolio bonds: The ratio of bonds to total assets in the fund

Fundamentals (four features):

1. price earnings: The ratio of price per share to earnings per share
2. price book: The ratio of price per share to book value per share
3. price sales: The ratio of price per share to sales per share
4. price cashflow: The ratio of price per share to cash flow per share

Sector weights (for 11 sectors) provided as percentages:

1. basic materials
2. consumer cyclical
3. financial services
4. real estate
5. consumer defensive
6. healthcare
7. utilities
8. communication services
9. energy
10. industrials
11. technology

Labels

Winning and losing ETFs or MFs are determined based on whether their returns are one standard deviation or more above or below the distribution of one-month fund returns across all ETFs or across all MFs, respectively. More precisely, the labels are:

1, if fund_return_1 month ≥ mean(fund_return_1 month) + one std.dev(-fund_return_1 month), indicating a winning fund;

-1, if fund_return_1 month ≤ mean(fund_return_1 month) – one std. dev(fund_return_1 month), indicating a losing fund; and

0, otherwise.

*Feature appears in the ETF dataset only.

**Feature appears in the MF dataset only.

Data sources: Kaggle, Yahoo Finance on 15 February 2019.

Methodology

The classification model is trained to determine whether a fund's performance is one standard deviation or more above the mean return (Label 1), within one standard deviation of the mean return (Label 0), or one standard deviation or more below the

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mean return (Label -1), where the mean return and standard deviation are either for all ETFs or all MFs, depending on the particular fund's type (ETF or MF). Performance is based on the one-month return of each fund as of 15 February 2019.

This procedure results in most of the funds being labeled as "0" (or average). After removing missing values in the dataset, there are 1,594 and 6,085 observations in the ETF and MF datasets, respectively. The data table is a 7,679 × 22 matrix, with 7,679 rows for each fund observation (1,594 for ETFs and 6,085 for MFs) and 22 columns for the 21 features plus the return label, and all data are recorded as of 15 February 2019.

The aim of the experiment is to identify not only winning and losing funds but also the features which are useful for distinguishing winners from losers. An important caveat, however, is that no claim is made that such features are causal.

A separate multi-classifier, with three classes, is run for each dataset. Four types of machine learning algorithms are used to build each classifier: (i) CART, (ii) SVM, (iii) KNN, and (iv) random forest. Random forest is an example of an ensemble method (based on bagging), whereas the other three algorithms do not use bagging.

A typical experimental design would involve using 70% of the data for training and holding 15% for tuning model hyperparameters and the remaining 15% of the data for testing. For simplicity, we shall not tune the hyperparameters but simply use the default settings without attempting to fine tune each one for best performance. So, in this case, we do not withhold 15% of the data for validation but instead train the classifier on a random split of 70% of the dataset, with the remaining 30% of the dataset used for testing. Crucially, for fairness of evaluation, each algorithm is trained and tested on identical data: The same 70% of observations are used for training each algorithm, and the same 30% are used for testing each one. The most important hyperparameters and settings for the algorithms are shown in Exhibit 13.

Exhibit 13: Parameter Settings for the Four Machine Learning Classifiers

1. CART: maximum tree depth: 5 levels
2. SVM: cost parameter: 1.0
3. KNN: number of nearest neighbors: 4
4. Random forest: number of trees: 100; maximum tree depth: 20 levels

The choices of hyperparameter values for the four machine learning classifiers are supported by theory, academic research, practice, and experimentation to yield a satisfactory bias–variance trade-off. For SVM, the cost parameter is a penalty on the margin of the decision boundary. A large cost parameter forces the SVM to use a thin margin, whereas a smaller cost parameter widens the margin. For random forests, recall that this is an ensemble method which uses multiple decision trees to classify, typically by majority vote. Importantly, no claim is made that these choices of hyperparameters are universally optimal for any dataset.

Results

The results of each classifier are evaluated separately on the test portion of the ETF and MF datasets. The evaluation metrics used are based on Type I and Type II classification errors, where a Type I error is a false positive (FP) and a Type II error is a false negative (FN). Correct classifications are true positive (TP) and true negative (TN).

- The first evaluation metric is **accuracy**, the percentage of correctly predicted classes out of total predictions. So, high accuracy implies low Type I and Type II errors.

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- **F1 score**, the second evaluation metric, is the weighted average of precision and recall. Precision is the ratio of correctly predicted positive classes to all predicted positive classes, and recall is the ratio of correctly predicted positive classes to all actual positive classes.

F1 score is a more appropriate evaluation metric to use than accuracy when there is unequal class distribution ("class imbalance") in the dataset, as is the case here. As mentioned, most of the funds in the ETF and MF datasets are designated as "0," indicating average performers.

Exhibit 14 shows the comparative performance results for each algorithm applied to the ETF dataset. These results show the random forest model is the most accurate (0.812), but once class imbalance is accounted for using F1 score (0.770), random forest is about as good as CART. Generally, ensemble methods, such as random forest, are expected to be at least as good as their single-model counterparts because ensemble forecasts generalize better out-of-sample. Importantly, while the relative accuracies and F1 scores across the different methods provide a basis for comparison, they do not speak to the absolute performance. In this regard, values approaching 1 suggest an excellent model, whereas values of approximately 1/3 would indicate the model is useless: 1/3 is premised on three (+1, 0, -1) equally distributed labels. However, because the distribution of classes is often not balanced, this ratio typically requires some adjustment.

Exhibit 14: Comparison of Accuracy and F1 Score for Each Classifier Applied to the ETF Dataset

	CART	SVM	KNN	Random Forest
Accuracy	0.770	0.774	0.724	0.812
F1 score	0.769	0.693	0.683	0.770

Exhibit 15 shows that the random forest model outperforms all the other classifiers under both metrics when applied to the MF dataset. Overall, the accuracy and F1 score for the SVM and KNN methods are similar for each dataset, and these algorithms are dominated by CART and random forest, especially in the larger MF dataset. The difference in performance between the two datasets for all the algorithms is to be expected, since the MF dataset is approximately four times larger than the ETF dataset and a larger sample set generally leads to better model performance. Moreover, the precise explanation of why random forest and CART outperform SVM and KNN is beyond the scope of this case. Suffice it to say that random forests are well known to be more robust to noise than most other classifiers.

Exhibit 15: Comparison of Accuracy and F1 Score for Each Classifier Applied to the Mutual Fund Dataset

	CART	SVM	KNN	Random Forest
Accuracy	0.959	0.859	0.856	0.969
F1 score	0.959	0.847	0.855	0.969

Exhibit 16 presents results on the relative importance of the features in the random forest model for both the ETF (Panel A) and MF (Panel B) datasets. Relative importance is determined by **information gain**, which quantifies the amount of information that the feature holds about the response. Information gain can be regarded as a form of

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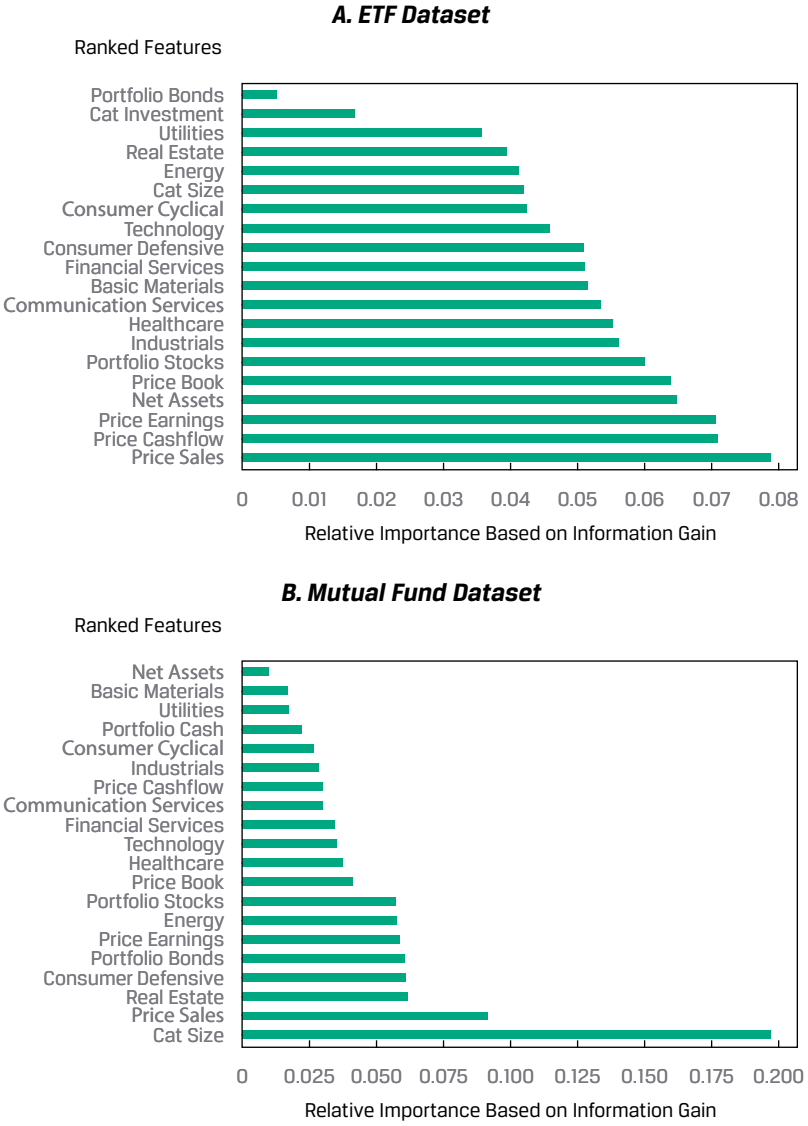
non-linear correlation between Y and X. Note the horizontal scale of Panel B (MF dataset) is more than twice as large as that of Panel A (ETF dataset), and the bar colors represent the feature rankings, not the features themselves.

Exhibit 16: Relative Importance of Features in the Random Forest Model



The prices-to-sales (price_sales) and prices-to-earnings (price_earnings) ratios are observed to be important indicators of performance, at about 0.08–0.09 and 0.06–0.07, respectively, in the random forest models for each dataset. The ratio of stocks to total assets (portfolio_stocks), at 0.06, is another key feature. Moreover, the industrials, health care, and communication services sector weightings are relatively important in the ETF dataset, while the real estate, consumer defensive, and energy sector weightings are key features in the MF dataset for differentiating between winning and losing funds.

Another important observation is that the category of the fund size (cat_size) is by far the most important feature in the model's performance for the MF dataset (≈ 0.20), whereas it is of much less importance for model performance using the ETF

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dataset (≈ 0.04). Conversely, net assets is a relatively important feature for model performance using the ETF dataset (0.065), while it is the least important feature when the random forest model is applied to the MF dataset (0.01).

Conclusion

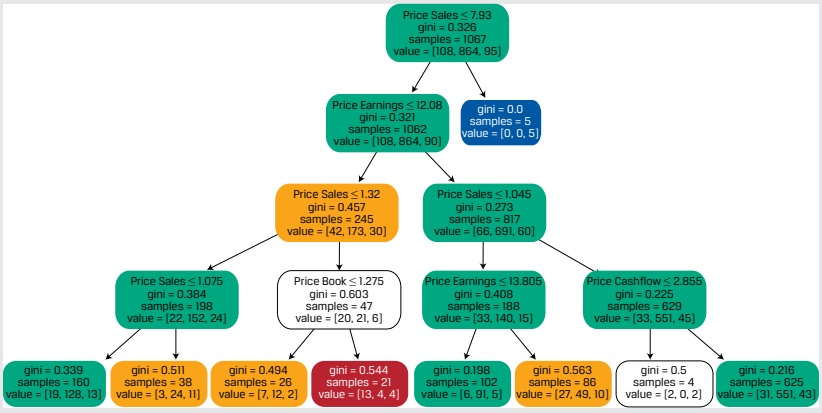
The research analyst has trained and tested machine learning–based models that she can use to identify potential winning and losing ETFs and MFs. Her classification models use input features based on fund type and size, asset class composition, fundamentals, and sector composition characteristics. She is more confident in her assessment of MFs than of ETFs, owing to the substantially larger sample size of the former. She is also confident that any imbalance in class has not led to misinterpretation of her models' results, since she uses F1 score as her primary model evaluation metric. Moreover, she determines that the best performing model using both datasets is an ensemble-type random forest model. Finally, she concludes that while fundamental ratios, asset class ratios, and sector composition are important features for both models, net assets and category size also figure prominently in discriminating between winning and losing ETFs and MFs.

EXAMPLE 5

Classification of Funds

The research analyst from the previous case uses CART to generate the decision tree shown in Exhibit 17, which she will use to predict whether and explain why a new ETF is likely to be a winner (+1), an average performer (0), or a loser (-1). This ETF's fundamental valuation ratios are as follows: Price-to-sales = 2.29, price-to-earnings = 7.20, price-to-book = 1.41, and price-to-cash flow = 2.65. Note that the sample size is 1,067 ETFs and the CART model uses just valuation ratios, because these are deemed the most important features for ETF performance classification.

Exhibit 17: CART-Based Decision Tree for EFT Performance Classification



Legend:

Darkest shade, 5th (last) level: Winner (Class = +1)

Light to medium shade: Average Performer (Class = 0); note that the medium shade indicates more confidence in the classification.

Darkest shade, 2nd level: Loser (Class = -1)

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White: Inconclusive, either because there is a tie with multiple categories or there are too few samples

Value: The number of sample cases in each of the three classes: Winner, Average Performer, or Loser

Path: Left path is True and right path is False.

1. Explain the CART model's prediction for performance of the new ETF: winner, loser, or average performer.

Solution:

Based on its valuation ratios (P/S = 2.29; P/E = 7.20; P/B = 1.41), the new ETF is predicted to be a winner because the decision path leads to the dark shaded, 5th level ("winner") terminal node. The split criteria and decisions are as follows:

Initial node: P/S ≤ 7.93 and EFT P/S = 2.29, so True.
2nd-level node: P/E ≤ 12.08 and EFT P/E = 7.20, so True.
3rd-level node: P/S ≤ 1.32 and EFT P/S = 2.29, so False.
4th-level node: P/B ≤ 1.275 and EFT P/B = 1.41, so False.
5th-level (terminal) node: darkest shaded terminal node indicates "winner."

2. Calculate the probability that the fund will be in the class predicted by the CART model.

Solution:

The output from the CART model in the darkest shaded, 5th level (winner) terminal node is [13, 4, 4], which indicates it includes 13 funds of Class +1 (winners), 4 funds of Class 0 (average performers), and 4 funds of Class -1 (losers). Thus, the probability predicted by the CART model that this ETF will be in the "winner" class is 13/21, or 62%. There are also equal probabilities of it being an average performer (19%) or a loser (19%).

3. Explain why the analyst should be cautious in basing the ETF's predicted performance solely on the CART-generated decision tree.

Solution:

There are several reasons why the analyst should be cautious in basing the ETF's predicted performance solely on the CART-generated decision tree. First, this CART model had a maximum depth of just five levels. Truncating at five levels facilitates visualization, but a more realistic decision path is likely to be nuanced and so would require greater depth. Second, only some of the important variables (from Exhibit 16) were used in generating this tree, again for simplicity of visualization. A CART model using additional features, including fund asset class ratios, sector composition, and, especially, net assets would be expected to generate a more accurate (using F1 score) model. Finally, the number of funds reaching the darkest shaded, 5th level ("winner") terminal node (21) is small compared to the total sample size (1,067), so there may be too few clear winners (13) under this decision path from which to draw a statistically significant conclusion. Besides increasing the maximum tree depth and adding more features, another approach the analyst might take in this case for achieving a more accurate model is random forest; being an ensemble classifier, a random forest model would generalize out-of-sample better than any single CART model.

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ESG DATA AS ALTERNATIVE DATA AND ML/AI FOR INTEGRATING ESG DATA INTO INVESTMENT DECISIONS

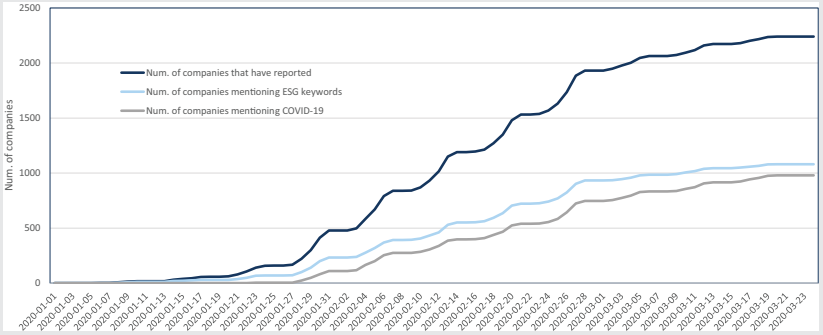
As an investment professional, how might you set about measuring the potential impact of climate change on a company's future prospects? Negative climate outcomes in coming years may include higher temperatures, more intense storms, melting glaciers, rising sea levels, shifting agricultural patterns, pressure on food and water, and new threats to human health. Assessing the likely severity of these future events and then quantifying the impact on companies is no easy task. Big Data techniques could be pivotal in generating usable information that could help investment professionals unlock long-term shareholder value.

Some fund managers, influenced by evolving investor preferences and increasing disclosure by companies on non-financial issues, have already incorporated ESG analysis into their investment processes. Governance ("G") data are generally objective: Investors are able to observe and measure corporate board actions, making governance comparable across companies and regions. Data on Environmental ("E") and Social ("S") impacts on listed companies, on the other hand, are more subjective, less reliable, and less comparable.

ESG data resemble alternative data in the sense that they have generally been poorly defined, are complex and unstructured, and need considerable due diligence before being used in investment decision making. Applying Machine Learning (ML) and Artificial Intelligence (AI) techniques can transform ESG data into meaningful information that is more useful for investment analysis.

Corporate sustainability reports often suffer from haphazard data collection and missing values. Equally, when data vendors acquire and combine raw ESG data into aggregate ESG scores, potential signals may be lost. ESG data and scoring across companies and data vendors can lack consistency and comparability; as a result, using simple summary scores in investment analysis is potentially flawed. Data analysts can apply data-science methods, such as data cleansing and data wrangling, to raw ESG data to create a structured dataset. Then, ML/AI techniques, such as natural language processing (NLP), can be applied to text-based, video, or audio ESG data. The foundation of NLP consists of supervised machine learning algorithms that typically include logistic regression, SVM, CART, random forests, or neural networks.

NLP can, for instance, search for key ESG words in corporate earnings calls. An increase in the number of mentions of, say, "human capital," employee "health and safety," or "flexible working" arrangements may indicate an increased focus on the "S" pillar of ESG. This would potentially raise the overall ESG score of a particular company. The results of such an application of NLP to corporate earnings calls are illustrated in the following exhibit:



Source: "GS SUSTAIN: ESG—Neither Gone Nor Forgotten" by Evan Tylenda, Sharmini Chetwode, and Derek R. Bingham, Goldman Sachs Global Investment Research (2 April 2020).

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ML/AI can help fund managers apply only those ESG factors that are relevant to a company and its sector. For example, "E" factors are important for mining and utility companies but less so for clothing manufacturers. Likewise, "S" factors are important for the global clothing manufacturing sector but less so for mining and utility companies.

ML/AI techniques are not used in isolation. ESG scoring systems tend to rely on cross-functional teams, with data scientists operating in tandem with economists, fundamental analysts, and portfolio managers to identify strengths and weaknesses of companies and sectors. Fundamental analysts, for instance, typically do not need to know the details of ML algorithms to make valuable contributions to the ESG investment workflow. The industry-specific knowledge of fundamental analysts can provide nuanced viewpoints that help to: 1) identify relevant raw data; 2) enable data scientists to incorporate ESG data into appropriate investment models; and 3) interpret model outputs and investment implications.

10 UNSUPERVISED ML ALGORITHMS AND PRINCIPAL COMPONENT ANALYSIS

- ☐ describe unsupervised machine learning algorithms—including principal components analysis, k-means clustering, and hierarchical clustering—and determine the problems for which they are best suited

Unsupervised learning is machine learning that does not use labeled data (i.e., no target variable); thus, the algorithms are tasked with finding patterns within the data themselves. The two main types of unsupervised ML algorithms shown in Exhibit 2 are dimension reduction, using principal components analysis, and clustering, which includes *k*-means and hierarchical clustering. These will now be described in turn.

Principal Components Analysis

Dimension reduction is an important type of unsupervised learning that is used widely in practice. When many features are in a dataset, representing the data visually or fitting models to the data may become extremely complex and "noisy" in the sense of reflecting random influences specific to a dataset. In such cases, dimension reduction may be necessary. Dimension reduction aims to represent a dataset with many typically correlated features by a smaller set of features that still does well in describing the data.

A long-established statistical method for dimension reduction is **principal components analysis (PCA)**. PCA is used to summarize or transform highly correlated features of data into a few main, uncorrelated composite variables. A **composite variable** is a variable that combines two or more variables that are statistically strongly related to each other. Informally, PCA involves transforming the covariance matrix of the features and involves two key concepts: eigenvectors and eigenvalues. In the context of PCA, **eigenvectors** define new, mutually uncorrelated composite variables that are linear combinations of the original features. As a vector, an eigenvector also represents a direction. Associated with each eigenvector is an eigenvalue. An **eigenvalue** gives the proportion of total variance in the initial data that is explained by

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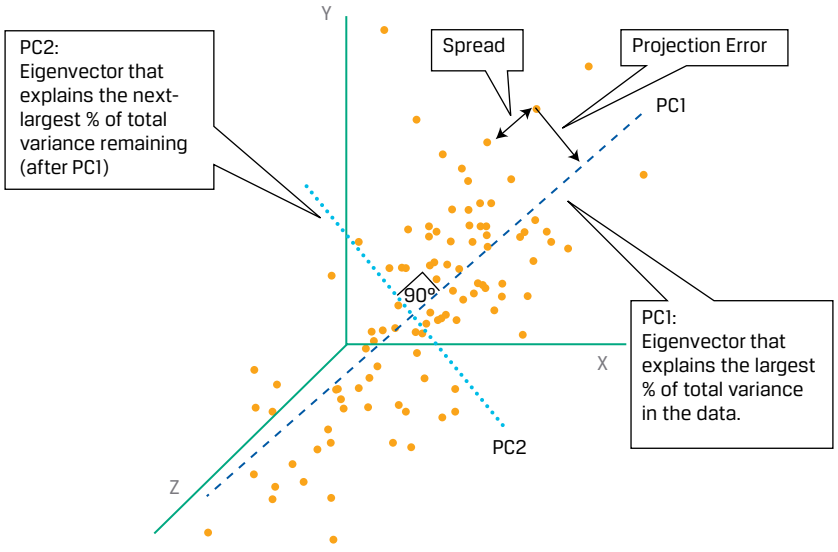
each eigenvector. The PCA algorithm orders the eigenvectors from highest to lowest according to their eigenvalues—that is, in terms of their usefulness in explaining the total variance in the initial data (this will be shown shortly using a scree plot). PCA selects as the first principal component the eigenvector that explains the largest proportion of variation in the dataset (the eigenvector with the largest eigenvalue). The second principal component explains the next-largest proportion of variation remaining after the first principal component; this process continues for the third, fourth, and subsequent principal components. Because the principal components are linear combinations of the initial feature set, only a few principal components are typically required to explain most of the total variance in the initial feature covariance matrix.

Exhibit 18 shows a hypothetical dataset with three features, so it is plotted in three dimensions along the *x*-, *y*-, and *z*-axes. Each data point has a measurement (*x*, *y*, *z*), and the data should be standardized so that the mean of each series (*x*'s, *y*'s, and *z*'s) is 0 and the standard deviation is 1. Assume PCA has been applied, revealing the first two principal components, PC1 and PC2. With respect to PC1, a perpendicular line dropped from each data point to PC1 shows the vertical distance between the data point and PC1, representing **projection error**. Moreover, the distance between each data point in the direction that is parallel to PC1 represents the spread or variation of the data along PC1. The PCA algorithm operates in such a way that it finds PC1 by selecting the line for which the sum of the projection errors for all data points is minimized and for which the sum of the spread between all the data is maximized. As a consequence of these selection criteria, PC1 is the unique vector that accounts for the largest proportion of the variance in the initial data. The next-largest portion of the remaining variance is best explained by PC2, which is at right angles to PC1 and thus is uncorrelated with PC1. The data points can now be represented by the first two principal components. This example demonstrates the effectiveness of the PCA algorithm in summarizing the variability of the data and the resulting dimension reduction.

Exhibit 18: First and Second Principal Components of a Hypothetical Three-Dimensional Dataset



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It is important to know how many principal components to retain because there is a trade-off between a lower-dimensional, more manageable view of a complex dataset when a few are selected and some loss of information. **Scree plots**, which show the proportion of total variance in the data explained by each principal component, can be helpful in this regard (see the accompanying sidebar). In practice, the smallest number of principal components that should be retained is that which the scree plot shows as explaining a desired proportion of total variance in the initial dataset (often 85% to 95%).

SCREE PLOTS FOR THE PRINCIPAL COMPONENTS OF RETURNS TO THE HYPOTHETICAL DLC 500 AND VLC 30 EQUITY INDEXES

In this illustration, researchers use scree plots and decide that three principal components are sufficient for explaining the returns to the hypothetical Diversified Large Cap (DLC) 500 and Very Large Cap (VLC) 30 equity indexes over the last 10-year period. The DLC 500 can be thought of as a diversified index of large-cap companies covering all economic sectors, while the VLC 30 is a more concentrated index of the 30 largest publicly traded companies. The dataset consists of index prices and more than 2,000 fundamental and technical features. Multi-collinearity among the features is a typical problem because that many features or combinations of features tend to have overlaps. To mitigate the problem, PCA can be used to capture the information and variance in the data. The following scree plots show that of the 20 principal components generated, the first 3 together explain about 90% and 86% of the variance in the value of the DLC 500 and VLC 30 indexes, respectively. The scree plots indicate that for each of these indexes, the incremental contribution to explaining the variance structure of the data is quite small after about the fifth principal component. Therefore, these less useful principal components can be ignored without much loss of information.

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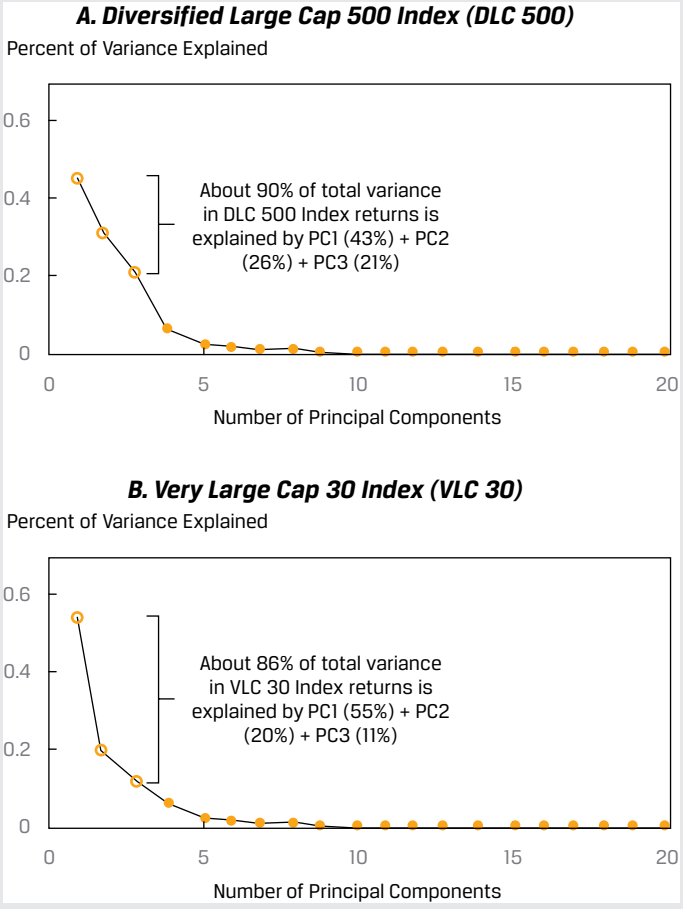
Scree Plots of Percent of Total Variance Explained by Each Principal Component for Hypothetical DLC 500 and VLC 30 Equity Indexes



The main drawback of PCA is that since the principal components are combinations of the dataset's initial features, they typically cannot be easily labeled or directly interpreted by the analyst. Compared to modeling data with variables that represent well-defined concepts, the end user of PCA may perceive PCA as something of a "black box."

Reducing the number of features to the most relevant predictors is very useful, even when working with datasets having as few as 10 or so features. Notably, dimension reduction facilitates visually representing the data in two or three dimensions. It is typically performed as part of exploratory data analysis, before training another supervised or unsupervised learning model. Machine learning models are quicker to train, tend to reduce overfitting (by avoiding the curse of dimensionality), and are easier to interpret if provided with lower-dimensional datasets.

CLUSTERING

11

- ☐ describe unsupervised machine learning algorithms—including principal components analysis, k-means clustering, and hierarchical clustering—and determine the problems for which they are best suited

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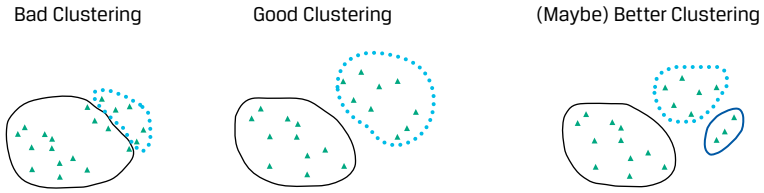
Clustering is another type of unsupervised machine learning, which is used to organize data points into similar groups called clusters. A **cluster** contains a subset of observations from the dataset such that all the observations within the same cluster are deemed "similar." The aim is to find a good clustering of the data—meaning that the observations inside each cluster are similar or close to each other (a property known as cohesion) and the observations in two different clusters are as far away from one another or are as dissimilar as possible (a property known as separation). Exhibit 19 depicts this intra-cluster cohesion and inter-cluster separation.

Exhibit 19: Evaluating Clustering—Intra-Cluster Cohesion and Inter-Cluster Separation



Clustering algorithms are particularly useful in the many investment problems and applications in which the concept of similarity is important. Applied to grouping companies, for example, clustering may uncover important similarities and differences among companies that are not captured by standard classifications of companies by industry and sector. In portfolio management, clustering methods have been used for improving portfolio diversification.

In practice, expert human judgment has a role in using clustering algorithms. In the first place, one must establish what it means to be "similar." Each company can be considered an observation with multiple features, including such financial statement items as total revenue and profit to shareholders, a wide array of financial ratios, or any other potential model inputs. Based on these features, a measure of similarity or "distance" between two observations (i.e., companies) can be defined. The smaller the distance, the more similar the observations; the larger the distance, the more dissimilar the observations.

A commonly used definition of distance is the Euclidian distance, the straight-line distance between two points. A closely related distance useful in portfolio diversification is correlation, which is the average Euclidian distance between a set of standardized points. Roughly a dozen different distance measures are used regularly in ML. In practice, the choice of the distance measures depends on the nature of the data (numerical or not) and the business problem being investigated. Once the relevant distance measure is defined, similar observations can be grouped together. We now introduce two of the more popular clustering approaches: *k*-means and hierarchical clustering.

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K-MEANS CLUSTERING

12

- ☐ describe unsupervised machine learning algorithms—including principal components analysis, k-means clustering, and hierarchical clustering—and determine the problems for which they are best suited

***K*-means** is an algorithm that repeatedly partitions observations into a fixed number, k, of non-overlapping clusters. The number of clusters, k, is a model hyperparameter. Each cluster is characterized by its **centroid** (i.e., center), and each observation is assigned by the algorithm to the cluster with the centroid to which that observation is closest. Notably, once the clusters are formed, there is no defined relationship between them.

The *k*-means algorithm follows an iterative process. It is illustrated in Exhibit 20 for $k = 3$ and a set of observations on a variable that can be described by two features. In Exhibit 20, the horizontal and vertical axes represent, respectively, the first and second features. For example, an investment analyst may want to group a set of firms into three groups according to two numerical measures of management quality. The algorithm groups the observations in the following steps:

1. *K*-means starts by determining the position of the k (here, 3) initial random centroids.
2. The algorithm then analyzes the features for each observation. Based on the distance measure that is used, *k*-means assigns each observation to its closest centroid, which defines a cluster.
3. Using the observations within each cluster, *k*-means then calculates the new (k) centroids for each cluster, where the centroid is the average value of their assigned observations.
4. *K*-means then reassigns the observations to the new centroids, redefining the clusters in terms of included and excluded observations.
5. The process of recalculating the new (k) centroids for each cluster is reiterated.
6. *K*-means then reassigns the observations to the revised centroids, again redefining the clusters in terms of observations that are included and excluded.

The *k*-means algorithm will continue to iterate until no observation is reassigned to a new cluster (i.e., no need to recalculate new centroids). The algorithm has then converged and reveals the final k clusters with their member observations. The *k*-means algorithm has minimized intra-cluster distance (thereby maximizing cohesion) and has maximized inter-cluster distance (thereby maximizing separation) under the constraint that $k = 3$.

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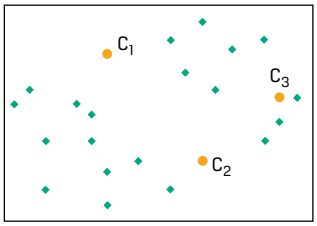
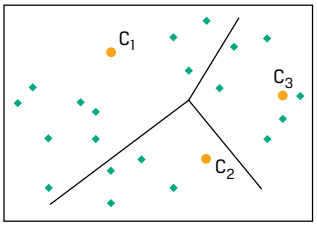
Exhibit 20: Example of 3-Means Algorithm

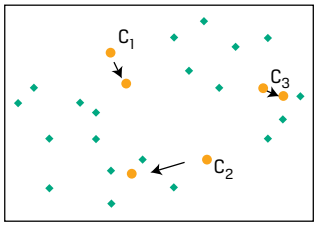
A. Chooses Initial Random Centroids: C1, C2, C3



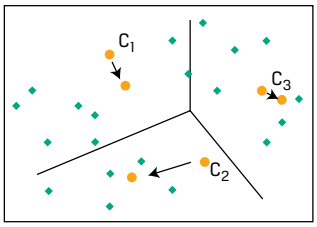
B. Assigns Each Obsevation to Nearest Centroid (defining initial 3 clusters)



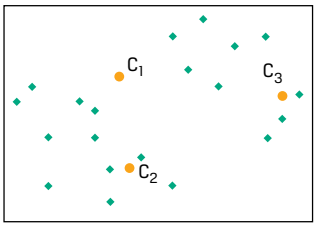
C. Calculates New Centroids as the Average Values of Observations in a Cluster



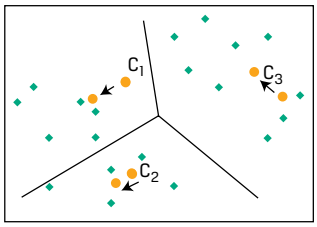
D. Reassigns Each Observation to the Nearest Centroid (from C)



E. Reiterates the Process of Recalculating New Centroids



F. Reassigns Each Observation to the Nearest Centroid (from E), Completing Second Iteration



The *k*-means algorithm is fast and works well on very large datasets, those with hundreds of millions of observations. However, the final assignment of observations to clusters can depend on the initial location of the centroids. To address this problem, the algorithm can be run several times using different sets of initial centroids, and then one can choose the clustering that is most useful given the business purpose.

One limitation of this technique is that the hyperparameter, *k*, the number of clusters in which to partition the data, must be decided before *k*-means can be run. So, one needs to have a sense of how many clusters are reasonable for the problem under investigation and the dataset being analyzed. Alternatively, one can run the algorithm using a range of values for *k* to find the optimal number of clusters—the *k* that minimizes intra-cluster distance and thereby maximizes intra-cluster similarity (i.e., cohesion) and that maximizes inter-cluster distance (i.e., separation). However, note that the final results can be subjective and dependent on the context of the problem and the particular training set. In practice, it is common to make the final choice of *k* based on face validity, such that the clusters feel sensible and are interpretable. This decision is greatly assisted by using summary information about the centroids and ranges of values and naming example items in each cluster.

For example, consider the Russell 3000 Index, which tracks the 3,000 highest market capitalization stocks in the United States. These 3,000 stocks can be grouped in 10, 50, or even more clusters based on their financial characteristics (e.g., total assets, total revenue, profitability, leverage) and operating characteristics (e.g., employee headcount, R&D intensity). Because companies in the same standard industry classification can have very different financial and operating characteristics, using *k*-means to derive different clusters can provide insights and understanding into the nature of "peer" groups. As mentioned, the exact choice of the *k*, the number of clusters, will depend on the level of precision or segmentation desired. In a similar vein, clustering

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can be used to classify collective investment vehicles or hedge funds as an alternative to standard classifications. Clustering analysis can also help visualize the data and facilitate detecting trends or outliers.

In sum, the *k*-means algorithm is among the most used algorithms in investment practice, particularly in data exploration for discovering patterns in high-dimensional data or as a method for deriving alternatives to existing static industry classifications.

HIERARCHICAL CLUSTERING

13

- describe unsupervised machine learning algorithms—including principal components analysis, k-means clustering, and hierarchical clustering—and determine the problems for which they are best suited

Hierarchical clustering is an iterative procedure used to build a hierarchy of clusters. In *k*-means clustering, the algorithm segments the data into a predetermined number of clusters; there is no defined relationship among the resulting clusters. In hierarchical clustering, however, the algorithms create intermediate rounds of clusters of increasing (in "agglomerative") or decreasing (in "divisive") size until a final clustering is reached. The process creates relationships among the rounds of clusters, as the word "hierarchical" suggests. Although more computationally intensive than *k*-means clustering, hierarchical clustering has the advantage of allowing the investment analyst to examine alternative segmentations of data of different granularity before deciding which one to use.

Agglomerative clustering (or bottom-up hierarchical clustering) begins with each observation being treated as its own cluster. Then, the algorithm finds the two closest clusters, defined by some measure of distance (similarity), and combines them into one new larger cluster. This process is repeated iteratively until all observations are clumped into a single cluster. A hypothetical example of how agglomerative clustering develops a hierarchical clustering scheme is depicted in the top part of Exhibit 21, where observations are lettered (A to K) and circles around observations denote clusters. The process begins with 11 individual clusters and then generates a sequence of groupings. The first sequence includes five clusters with two observations each and one cluster with a single observation, G, for a total of six clusters. It then generates two clusters—one cluster with six observations and the other with five observations. The final result is one large cluster containing all 11 observations. It is easily seen that this final large cluster includes the two main sub-clusters, with each containing three smaller sub-clusters.

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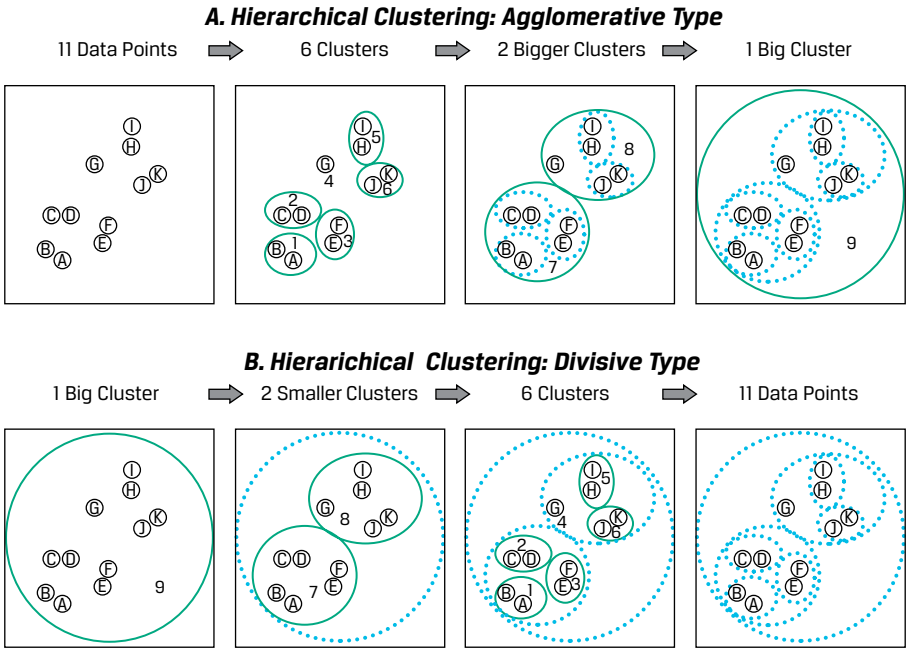
Exhibit 21: Agglomerative and Divisive Hierarchical Clustering



By contrast, **divisive clustering** (or top-down hierarchical clustering) starts with all the observations belonging to a single cluster. The observations are then divided into two clusters based on some measure of distance (similarity). The algorithm then progressively partitions the intermediate clusters into smaller clusters until each cluster contains only one observation. Divisive clustering is depicted in the bottom part of Exhibit 21, which begins with all 11 observations in one large cluster. Next, the algorithm generates two smaller clusters, one with six observations and the other with five observations, and then six clusters, with two observations each except for observation G, which is its own cluster. Finally, 11 clusters are generated, with each cluster containing only one observation.

Although this is not a typical outcome (because the two methods generally use different algorithms), in this hypothetical illustration, the agglomerative and divisive clustering methods produced the same result: two main sub-clusters each having three smaller sub-clusters. The analyst could decide between using a six- or a two-cluster representation of the data. The agglomerative method is the approach typically used with large datasets because of the algorithm's fast computing speed. The agglomerative clustering algorithm makes clustering decisions based on local patterns without initially accounting for the global structure of the data. As such, the agglomerative method is well suited for identifying small clusters. However, because the divisive method starts with a holistic representation of the data, the divisive clustering algorithm is designed to account for the global structure of the data and thus is better suited for identifying large clusters.

To decide on the closest clusters for combining in the agglomerative process or for dividing in the divisive process, an explicit definition for the distance between two clusters is required. Some commonly used definitions for the distance between two clusters involve finding the minimum, the maximum, or the average of the straight-line distances between all the pairs of observations in each cluster.

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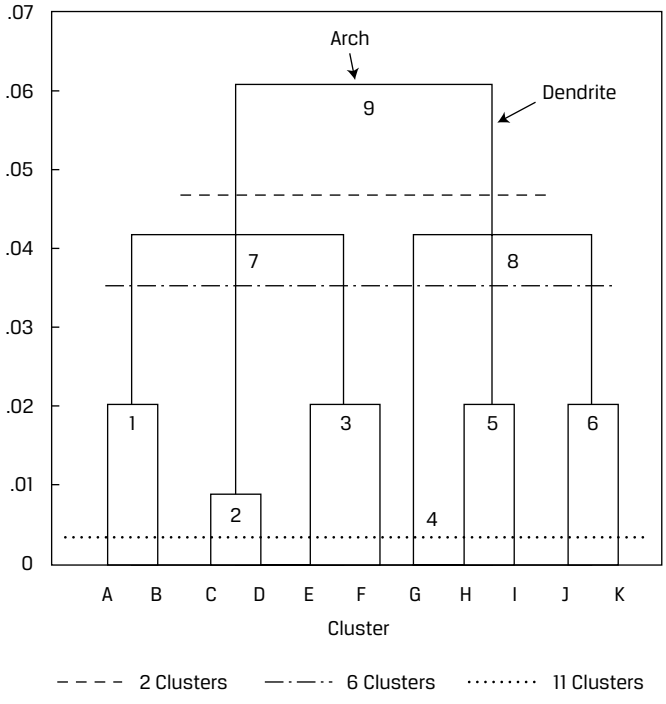
Dendrograms

A type of tree diagram for visualizing a hierarchical cluster analysis is known as a **dendrogram**, which highlights the hierarchical relationships among the clusters. Exhibit 22 shows a dendrogram representation for the clustering shown in Exhibit 21. First, a few technical points on dendrograms bear mentioning—although they may not all be apparent in Exhibit 22. The *x*-axis shows the clusters, and the *y*-axis indicates some distance measure. Clusters are represented by a horizontal line, the arch, which connects two vertical lines, called dendrites, where the height of each arch represents the distance between the two clusters being considered. Shorter dendrites represent a shorter distance (and greater similarity) between clusters. The horizontal dashed lines cutting across the dendrites show the number of clusters into which the data are split at each stage.

The agglomerative algorithm starts at the bottom of the dendrite, where each observation is its own cluster (A to K). Agglomerative clustering then generates the six larger clusters (1 to 6). For example, Clusters A and B combine to form Cluster 1, and Observation G remains its own cluster, now Cluster 4. Moving up the dendrogram, two larger clusters are formed, where, for example, Cluster 7 includes Clusters 1 to 3. Finally, at the top of the dendrogram is the single large cluster (9). The dendrogram readily shows how this largest cluster is composed of the two main sub-clusters (7 and 8), each having three smaller sub-clusters (1 to 3 and 4 to 6, respectively). The dendrogram also facilitates visualization of divisive clustering by starting at the top of the largest cluster and then working downward until the bottom is reached, where all 11 single-observation clusters are shown.

Exhibit 22: Dendrogram of Agglomerative Hierarchical Clustering



Clustering has many applications in investment management. For example, portfolio diversification can be approached as a clustering problem with the aim of optimally diversifying risks by investing in assets from multiple different clusters. Because the

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clusters have maximum inter-cluster separation, diversifying among them helps ensure that the portfolio reflects a wide diversity of characteristics with well-diversified risk. In contrast, information that investments are concentrated in a cluster indicates a high probability of concentrated risk. Finally, it is important to note that while the results of clustering algorithms are often difficult to evaluate (because the resulting clusters themselves are not explicitly defined), they are still very useful in practice for uncovering important underlying structure (namely, similarities among observations) in complex datasets.

EXAMPLE 6

Investment Uses of Clustering Algorithms

István Perényi is a portfolio manager of the Europe Diversified Equity Fund ("the Fund") within the Diversified Investment Management Company (DIMCO) fund family. The Fund is benchmarked to the STOXX Europe 600 Index, which spans 17 countries, 19 industry sectors, and three market capitalization groupings (large-, mid-, and small-cap).

Examining the Fund's most recent performance, Perényi is concerned that the Fund's holdings, although approximately aligned with the STOXX Europe 600 Index's country weights, may have unrecognized risk biases and concentrations. Perényi asks Elsa Lund, DIMCO's chief risk officer, to investigate the Fund's diversification. Lund asks her analysts for ideas on how Perényi's request can be addressed and receives three suggestions:

Suggestion 1	Estimate the country, industry, and market cap exposures of each Fund holding, aggregate them, and compare the aggregate exposures to the benchmark's exposures. Then, examine mismatches for evidence of unexpected biases or concentrations.
Suggestion 2	Identify inherent groupings among fund holdings based on a broad set of eight numerical (operating and financial) measures related to the holdings' characteristics. Then, examine the groupings for evidence of unexpected biases or concentrations.
Suggestion 3	Regress the return of the Fund on a set of country equity market indexes and sector indexes based on the Fund's benchmark. Then, examine the regression coefficients for evidence of unexpected biases or concentrations.

Lund has several questions for analyst Greg Kane about using one or more clustering machine learning algorithms in relation to addressing Perényi's request.

Lund asks whether any information needs to be specified for the ML clustering algorithms no matter which one is used. Kane replies that only the distance measure that the algorithm will use and the hyperparameter, k, for k-means clustering need to be specified.

Lund further asks whether there would be an advantage to using k-means clustering as opposed to hierarchical clustering. Kane replies that in his opinion, hierarchical clustering is the more appropriate algorithm.

1. Which analyst suggestion is *most likely* to be implemented using machine learning?

   A. Suggestion 1

   B. Suggestion 2

   C. Suggestion 3

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Solution:

B is correct. A machine learning clustering algorithm could be used to implement Suggestion 2. A and C are incorrect because Suggestions 1 and 3, respectively, can be addressed easily using traditional regression analysis.

2. Kane's reply to Lund's first question about specification of ML clustering models is:

   A. correct.

   B. not correct, because other hyperparameters must also be specified.

   C. not correct, because the feature set for describing the measure used to group holdings must also be specified.

Solution:

C is correct. Beyond specifying a distance measure and the *k* for *k*-means, whichever clustering algorithm is selected, the feature set used to group holdings by similarities must also be specified. Operating and financial characteristics of the companies represented in the Fund's portfolio are examples of such features.

3. The best justification for Kane's preference for hierarchical clustering in his reply to Lund's second question is that Kane is *most likely* giving consideration to:

   A. the speed of the algorithms.

   B. the dimensionality of the dataset.

   C. the need to specify the hyperparameter, *k*, in using a *k*-means algorithm.

Solution:

C is correct. The value of the hyperparameter, *k*, the number of distinct groups into which the STOXX Europe 600 Index can be segmented, is not known and needs to be specified in advance by the analyst. Using a hierarchical algorithm, the sorting of observations into clusters will occur without any prior input on the analyst's part.

CASE STUDY: CLUSTERING STOCKS BASED ON CO-MOVEMENT SIMILARITY

14

☐ describe unsupervised machine learning algorithms—including principal components analysis, k-means clustering, and hierarchical clustering—and determine the problems for which they are best suited

The following case study was developed and written by Matthew Dixon, PhD, FRM.

An endowment fund's Investment Committee is seeking three "buy" recommendations for the fund's large-cap equity portfolio. An analyst working for the Investment Committee is given a subset of eight stocks from the S&P 500 Index and asked to determine the co-movement similarity (i.e., correlation) of their returns. Specifically,

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for diversification purposes, the Investment Committee wants the correlation of returns between the recommended stocks to be low, so the analyst decides to use clustering to identify the most similar stocks and then choose one stock from each cluster. Although this case study focuses mainly on hierarchical agglomerative clustering, the analyst's results using other clustering algorithms (i.e., divisive clustering and *k*-means) are also briefly discussed. Exhibit 23 provides a description of the data used by the analyst.

Exhibit 23: Dataset of Eight Stocks from the S&P 500 Index

Description: Daily adjusted closing prices of eight S&P 500 member stocks

Trading Dates: 30 May 2017 to 24 May 2019

Number of Observations: 501

Stocks (Ticker Symbols): AAPL, F, FB, GM, GS, GOOG, JPM, and UBS

The following steps are taken by the analyst to perform the hierarchical agglomerative cluster analysis:

1. Collect panel data on adjusted closing prices for the stocks under investigation.
2. Calculate the daily log returns for each stock, where each time series of stock returns is an n-vector (n = 500).
3. Run the agglomerative hierarchical clustering algorithm.
   a. The algorithm calculates the pairwise distance (i.e., Euclidean distance) between vectors of any two stocks' returns. Each pairwise distance is an element of a distance matrix (i.e., dissimilarity matrix) with zero diagonals.
   b. The algorithm starts with each stock as its own cluster, finds the pair of clusters which are closest to each other, and then redefines them as a new cluster.
   c. The algorithm finds the distances from this new cluster to the remaining return clusters. Using a process called average (centroid) linkage, it determines the distances from the center of the new cluster to the centers of the remaining clusters. Note that there are several other linkage methods, but whichever method is selected, the algorithm proceeds in the same fashion: It combines the pair of clusters which are closest, redefines them as a new cluster, and recalculates the distances to the remaining clusters.
4. Repeat Step 3c until the data are aggregated into a single large cluster.
5. Plot the resulting dendrogram to visualize the hierarchical clusters and draw the highest horizontal line intersecting three (i.e., the desired number of clusters, since the Investment Committee wants three "buy" recommendations) vertical lines (or dendrites) to determine the appropriate cluster configuration.

Exhibit 24 shows for illustrative purposes a subset of the panel data on daily returns, calculated from the adjusted closing prices of the eight stocks collected in Step 1. The clustering is performed on the daily returns.

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Exhibit 24: Subset of Stock Returns, Calculated from Adjusted Closing Prices, for Clustering

Date	JPM	UBS	GS	FB	AAPL	GOOG	GM	F
2017-05-31	−0.021	−0.007	−0.033	−0.006	−0.006	−0.011	0.012	0.004
2017-06-01	0.011	0.013	0.018	0.000	0.003	0.002	0.015	0.026
2017-06-02	−0.005	−0.002	−0.008	0.014	0.015	0.009	0.001	−0.005
2017-06-05	0.002	−0.007	0.003	0.000	−0.010	0.008	0.000	−0.009
2017-06-06	0.002	0.002	0.003	−0.005	0.003	−0.007	−0.001	−0.012

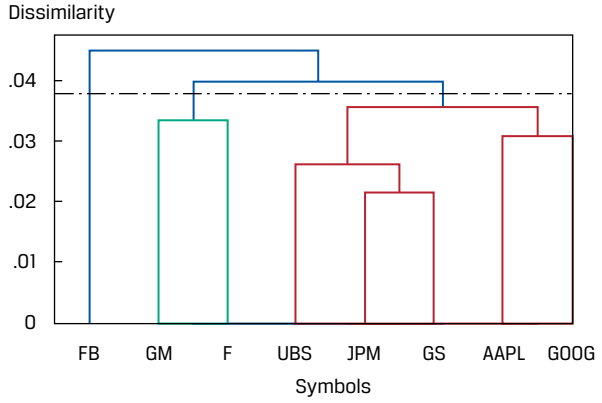
The results of the remaining steps are described using the distance matrix shown in Exhibit 25.

Exhibit 25: Distance Matrix for Hierarchical Agglomerative Clustering

	JPM	UBS	GS	FB	AAPL	GOOG	GM	F
JPM	0.000	0.243	0.215	0.456	0.364	0.332	0.358	0.348
UBS	0.243	0.000	0.281	0.460	0.380	0.338	0.384	0.385
GS	0.215	0.281	0.000	0.471	0.375	0.345	0.383	0.393
FB	0.456	0.460	0.471	0.000	0.437	0.357	0.491	0.480
AAPL	0.364	0.380	0.375	0.437	0.000	0.307	0.445	0.456
GOOG	0.332	0.338	0.345	0.357	0.307	0.000	0.405	0.422
GM	0.358	0.384	0.383	0.491	0.445	0.405	0.000	0.334
F	0.348	0.385	0.393	0.480	0.456	0.422	0.334	0.000

The distance matrix reveals the closest pair of stocks is JPM and GS, with a distance of 0.215. Therefore, this pair becomes the first combined cluster as shown in the dendrogram in Exhibit 26. Note that the vertical distance connecting the various clusters represents the Euclidean distance between clusters, so the arch between this pair has a height of 0.215. Now that JPM and GS are paired in a cluster (i.e., GS_JPM), we treat the mean of their two return vectors as a new point.

Exhibit 26: Dendrogram for Hierarchical Agglomerative Clustering



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From the distance matrix, the average distance of UBS to the new cluster (i.e., GS_JPM) is the sum of the distance between UBS and JPM, 0.243, and the distance between UBS and GS, 0.281, divided by two, which is 0.262 (= (0.243 + 0.281)/2). Since this distance is smaller than the distance between any of the other unpaired stock clusters, UBS is merged with this cluster to create a new cluster (i.e., GS_JPM_UBS). The height of the arch in the dendrogram for this new cluster is 0.262, which is now observed to contain three banking sector stocks. Although not shown in the dendrogram, the cluster is identified by the return vector averaged over the three stocks.

The next closest pair of points, whether stock to stock or stock to cluster, is AAPL and GOOG, with a distance of 0.307, so the algorithm merges these two points into a second cluster (i.e., AAPL_GOOG), with an arch height of 0.307. Next, GM and F are paired into a third cluster (i.e., F_GM), with an arch height of 0.334. Finally, the first two clusters are merged to form a five-stock cluster (i.e., GS_JPM_UBS_AAPL_GOOG), with an arch height of 0.356. Note that this value is determined by taking the average distance between the three banks and AAPL and GOOG: 0.356 = (0.364 + 0.380 + 0.375 + 0.332 + 0.338 + 0.345)/6. The result is three separate clusters: the five-stock cluster, F_GM, and FB by itself. Also, note the horizontal dashed line that cuts the dendrogram into three distinct clusters, with FB as its own cluster.

This agglomerative hierarchical clustering analysis reveals some interesting preliminary results—largely grouping the stocks by their sectors but also uncovering some anomalies. In particular, FB is found to behave quite differently, in terms of return co-movement similarity, from the other technology stocks (AAPL and GOOG). Also, AAPL and GOOG are found to behave more like the bank stocks and less like the auto stocks (F and GM), which appear in their own cluster.

In contrast to agglomerative clustering, the divisive clustering algorithm starts with all stocks assigned to one large cluster and then splits the cluster into sub-clusters recursively, until each stock occupies its own cluster. Determining how to split the first cluster requires searching over all combinations of possible splits, so it is too numerically intensive to cover the details here. However, results of the first two splits for divisive clustering, into three clusters, are shown in Exhibit 27. Results for *k*-means, with $k = 3$, and agglomerative clustering are also presented.

Exhibit 27: Comparison of Results of Different Clustering Algorithms

	Agglomerative	K-means	Divisive
AAPL	3	2	2
F	2	1	1
FB	1	2	3
GM	2	1	1
GOOG	3	2	2
GS	3	3	1

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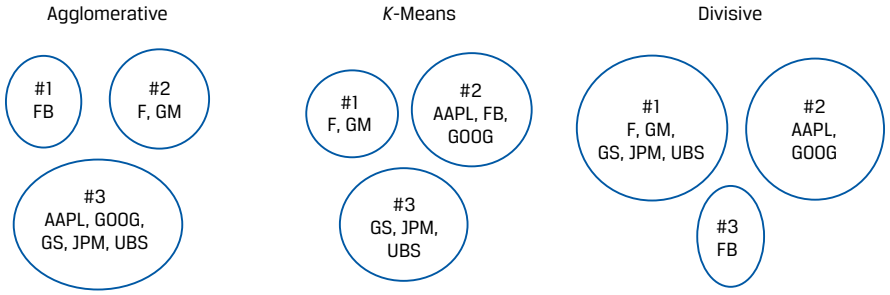
	Agglomerative	K-means	Divisive
JPM	3	3	1
UBS	3	3	1



Whereas the assignment of the cluster number (1, 2, 3), shown in the upper panel, can be taken as arbitrary across each algorithm, the useful information is in the grouping of like stocks. As seen in the stylized clusters in the lower panel, all three clustering algorithms agree that bank stocks belong in the same cluster. Both hierarchical agglomerative and *k*-means algorithms also agree that auto stocks belong in their own separate cluster. *K*-means clusters the stocks precisely by industry sector, whereas hierarchical agglomerative and divisive clustering identify FB as an outlier and place it in its own cluster. In general, the most agreement is expected between the two hierarchical clustering algorithms, although their results are not guaranteed to match, even when using the same linkage process. *K*-means starts with three clusters ($k = 3$) and iteratively swaps points in and out of these clusters using a partitioning mechanism different from that of hierarchical clustering. Thus, *k*-means results are typically not expected to match those of hierarchical clustering.

In conclusion, based on the analyses of the co-movement similarity of returns among the eight stocks using the agglomerative clustering algorithm and the Investment Committee's requirement that the correlation of returns between the recommended stocks should be low, the analyst's recommendation should be as follows:

- buy FB,
- buy the most attractive of the two auto stocks (F or GM), and
- buy the most attractive of the three bank stocks (GS, JPM, or UBS).

EXAMPLE 7

Hierarchical Agglomerative Clustering

Assume the analyst is given the same set of stocks as previously excluding F and GM (i.e., no auto stocks)—so now, six stocks. Using the information from this mini-case study, answer the following questions:

1. Describe how the inputs to the hierarchical agglomerative clustering algorithm would differ from those in the mini-case study.

Solution:

The panel data on closing prices and daily log returns would include the same stocks as before but without F and GM—so, AAPL, FB, GOOG, GS, JPM, and UBS. The distance matrix would also appear the same except without F, GM, or any of the pairwise distances between them and the remaining stocks.

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2. Describe the three clusters that would now result from running the hierarchical agglomerative clustering algorithm.

Solution:

The three clusters that would now result from running the agglomerative clustering algorithm are GS_JPM_UBS (i.e., one cluster of three bank stocks), AAPL_GOOG (i.e., one cluster of two technology stocks), and FB by itself.

3. Explain why these results differ from the previous case, with eight stocks (including the two auto stocks).

Solution:

The agglomerative clustering algorithm now combines GS and JPM and then UBS, as before, to form a bank cluster. Next, and as previously, the algorithm combines AAPL and GOOG into a cluster. However, without the auto stocks, there is no need to combine AAPL_GOOG with the bank cluster. There are now three distinct clusters, since (as before) the algorithm treats FB as its own cluster, given the high degree of return co-movement dissimilarity between FB and the other clusters (i.e., AAPL_GOOG, and GS_JPM_UBS).

4. Describe the analyst's new recommendation to the Investment Committee.

Solution:

The analyst's new recommendation to the Investment Committee would be to buy FB, buy the cheapest of AAPL or GOOG, and buy the most attractive of the three bank stocks (GS, JPM, or UBS).

15 NEURAL NETWORKS, DEEP LEARNING NETS, AND REINFORCEMENT LEARNING

☐ describe neural networks, deep learning nets, and reinforcement learning

The artificial intelligence revolution has been driven in large part by advances in neural networks, deep learning algorithms, and reinforcement learning. These sophisticated algorithms can address highly complex machine learning tasks, such as image classification, face recognition, speech recognition, and natural language processing. These complicated tasks are characterized by non-linearities and interactions between large numbers of feature inputs. We now provide an overview of these algorithms and their investment applications.

Neural Networks

Neural networks (also called artificial neural networks, or ANNs) are a highly flexible type of ML algorithm that have been successfully applied to a variety of tasks characterized by non-linearities and complex interactions among features. Neural networks

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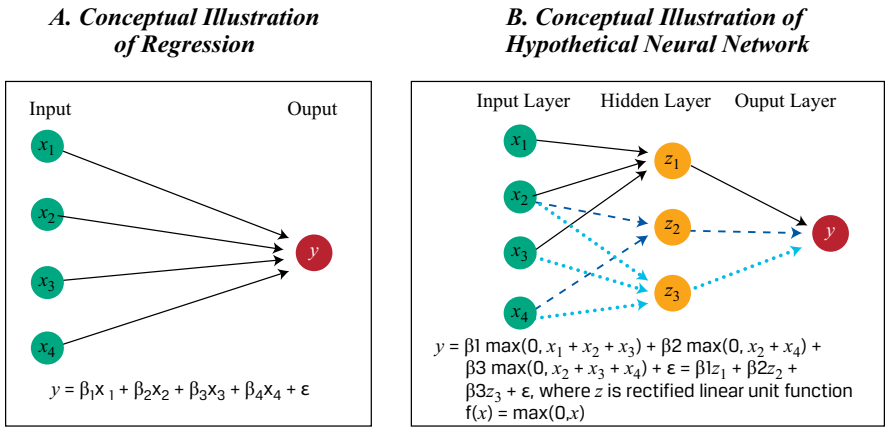
are commonly used for classification and regression in supervised learning but are also important in reinforcement learning, which does not require human-labeled training data.

Exhibit 28 shows the connection between multiple regression and neural networks. Panel A represents a hypothetical regression for data using four inputs, the features x_1 to x_4, and one output—the predicted value of the target variable y. Panel B shows a schematic representation of a basic neural network, which consists of nodes (circles) connected by links (arrows connecting nodes). Neural networks have three types of layers: an input layer (here with a node for each of the four features); hidden layers, where learning occurs in training and inputs are processed on trained nets; and an output layer (here consisting of a single node for the target variable y), which passes information outside the network.

Besides the network structure, another important difference between multiple regression and neural networks is that the nodes in the neural network's hidden layer transform the inputs in a non-linear fashion into new values that are then combined into the target value. For example, consider the popular rectified linear unit (ReLU) function, $f(x) = \max(0, x)$, which takes on a value of zero if there is a negative input and takes on the value of the input if it is positive. In this case, y will be equal to β_1 times z_1, where z_1 is the maximum of $(x_1 + x_2 + x_3)$ or 0, plus β_2 times z_2, the maximum of $(x_2 + x_4)$ or 0, plus β_3 times z_3, the maximum of $(x_2 + x_3 + x_4)$ or 0, plus an error term.

Exhibit 28: Regression and Neural Networks (Regression with Transformed Features)



Note that for neural networks, the feature inputs would be scaled (i.e., standardized) to account for differences in the units of the data. For example, if the inputs were positive numbers, each could be scaled by its maximum value so that their values lie between 0 and 1.

Exhibit 29 shows a more complex neural network, with an input layer consisting of four nodes (i.e., four features), one hidden layer consisting of five hidden nodes, and an output node. These three numbers—4, 5, and 1—for the neural network are hyperparameters that determine the structure of the neural network.

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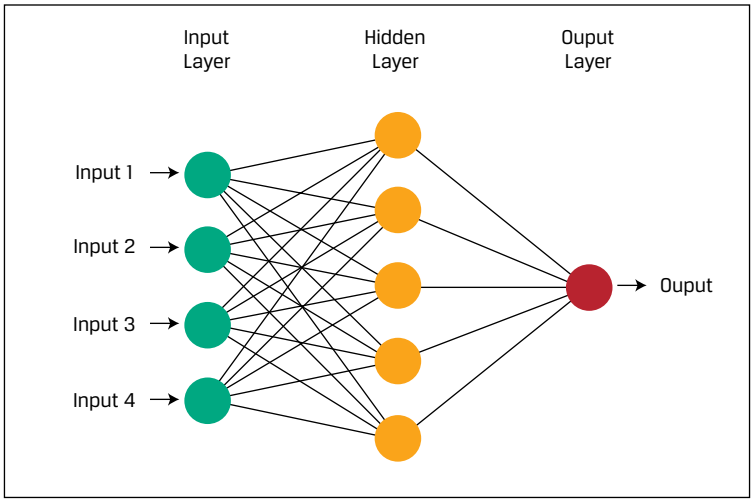
Exhibit 29: A More Complex Neural Network with One Hidden Layer



Now consider any of the nodes to the right of the input layer. These nodes are sometimes called "neurons" because they process information received. Take the topmost hidden node. Four links connect to that node from the inputs, so the node gets four values transmitted by the links. Each link has a weight meant to represent its importance (initially these weights may be assigned randomly). Each node has, conceptually, two functional parts: a summation operator and an activation function. Once the node receives the four input values, the **summation operator** multiplies each value by its respective weight and then sums the weighted values to form the total net input. The total net input is then passed to the **activation function**, which transforms this input into the final output of the node. Informally, the activation function operates like a light dimmer switch that decreases or increases the strength of the input. The activation function, which is chosen by the modeler (i.e., a hyperparameter), is characteristically non-linear, such as an S-shaped (sigmoidal) function (with output range of 0 to 1) or the rectified linear unit function shown in Panel B of Exhibit 28. Non-linearity implies that the rate of change of output differs at different levels of input.

This activation function is shown in Exhibit 30, where in the left graph a negative total net input is transformed via the S-shaped function into an output close to 0. This low output implies the node does not trigger, so there is nothing to pass to the next node. Conversely, in the right graph a positive total net input is transformed into an output close to 1, so the node does trigger. The output of the activation function is then transmitted to the next set of nodes if there is a second hidden layer or, as in this case, to the output layer node as the predicted value. The process of transmission just described (think of forward pointing arrows in Exhibit 29) is referred to as **forward propagation**.

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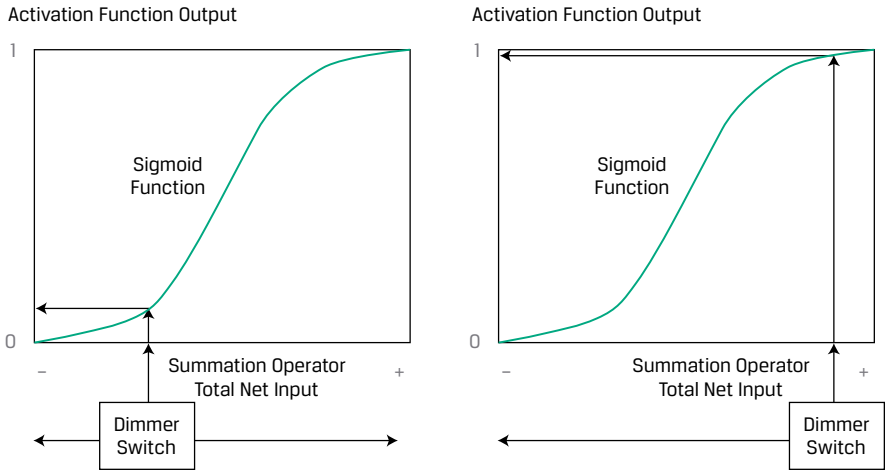
Exhibit 30: Activation Function as "Light Dimmer Switch" at Each Node in a Neural Network



Starting with an initialized set of random network weights (i.e., the weights assigned to each of the links), training a neural network in a supervised learning context is an iterative process in which predictions are compared to actual values of labeled data and evaluated by a specified performance measure (e.g., mean squared error). Then, the network weights are adjusted to reduce total error of the network. (If the process of adjustment works backward through the layers of the network, this process is called **backward propagation**). Learning takes place through this process of adjustment to the network weights with the aim of reducing total error. Without proliferating notation relating to nodes, the gist of the updating can be expressed informally as

New weight

= (Old weight) – (Learning rate) × (Partial derivative of the total error with respect to the old weight),

where partial derivative is a gradient or rate of change of the total error with respect to the change in the old weight and **learning rate** is a hyperparameter that affects the magnitude of adjustments. When learning is completed, all the network weights have assigned values; these are the parameters of the network.

The structure of a network in which all the features are interconnected with non-linear activation functions allows neural networks to uncover and approximate complex non-linear relationships among features. Broadly speaking, when more nodes and more hidden layers are specified, a neural network's ability to handle complexity tends to increase (but so does the risk of overfitting).

Asset pricing is a noisy, stochastic process with potentially unstable relationships that challenge modeling processes, so researchers are asking if machine learning can improve our understanding of how markets work. Research comparing statistical and machine learning methods' abilities to explain and predict equity prices so far indicates that simple neural networks produce models of equity returns at the individual stock and portfolio level that are superior to models built using traditional statistical methods due to their ability to capture dynamic and interacting variables. This suggests that ML-based models, such as neural networks, may simply be better able to cope with the non-linear relationships inherent in security prices. However, the trade-offs in using neural networks are their lack of interpretability (i.e., black box nature) and the large amounts of data and high computation intensity needed to train such models; thus, neural networks may not be a good choice in many investment applications.

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16 DEEP NEURAL NETWORKS

- describe neural networks, deep learning nets, and reinforcement learning

The previous discussion of neural networks was limited to types of neural networks referred to as "shallow neural networks"—exhibiting just one hidden layer. Neural networks with many hidden layers—at least 2 but potentially more than 20—are known as **deep neural networks** (DNNs). DNNs are the foundation of deep learning and have proven to be successful across a wide range of artificial intelligence applications. Advances in DNNs have driven developments in many complex activities, such as image, pattern, and speech recognition. To state the operation of DNNs succinctly, they take a set of inputs x from a feature set (the input layer), which are then passed to a layer of non-linear mathematical functions (neurons) with weights w_{ij} (for neuron i and input j), each of which usually produces a scaled number in the range (0, 1) or (–1, 1). These numbers are then passed to another layer of functions and into another and so on until the final layer produces a set of probabilities of the observation being in any of the target categories (each represented by a node in the output layer). The DNN assigns the category based on the category with the highest probability. The DNN is trained on large datasets; during training, the weights, w_i, are determined to minimize a specified loss function.

In practice, while the number of nodes in the input and the output layers are typically determined by the characteristics of the features and predicted output, many model hyperparameters still must be decided, particularly the number of hidden layers, the number of nodes per hidden layer, and their connectivity and activation architecture. The objective is to choose them to achieve the best out-of-sample performance, but it is still a challenge with no simple solution. As such, a good starting point is a "reasonable" guess for hyperparameters based on experience and literature. The researcher can then observe the result and adjust the hyperparameters incrementally until the model performance goal is reached. In practice, DNNs require substantial time to train, and systematically varying the hyperparameters may not be feasible. So, for many problems with relatively small datasets, one can start with just two or three hidden layers and a few hundred nodes before tuning the parameters until a model with acceptable predictive power is achieved.

DNNs have been shown to be useful in general for pattern recognition problems (e.g., character and image recognition), credit card fraud detection, vision and control problems in autonomous cars, natural language processing (such as machine translation), and other applications. DNNs have become hugely successful because of a confluence of three developments: (1) the availability of large quantities of machine-readable data to train models, (2) advances in analytical methods for fitting these models, and (3) fast computers, especially new chips in the graphics processing unit (GPU) class, tailored for the type of calculations done on DNNs.

Several financial firms are experimenting with DNNs for trading as well as automating their internal processes. Culkin and Das (2017) described how they trained DNNs to price options, mimicking the Black–Scholes–Merton model. Their research used the same six input parameters for the model as input layer features—spot price, strike, time to maturity, dividend yield, risk-free interest rate, and volatility—with four hidden layers of 100 neurons each and one output layer. The predicted option prices out-of-sample were very close to the actual option prices: A regression of predicted option prices on actual prices had an R^2 of 99.8%.

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Reinforcement Learning

Reinforcement learning (RL) made headlines in 2017 when DeepMind's AlphaGo program beat the reigning world champion at the ancient game of Go. The RL framework involves an agent that is designed to perform actions that will maximize its rewards over time, taking into consideration the constraints of its environment. In the case of AlphaGo, a virtual gamer (the agent) uses his or her console commands (the actions) with the information on the screen (the environment) to maximize his or her score (the reward). Unlike supervised learning, reinforcement learning has neither direct labeled data for each observation nor instantaneous feedback. With RL, the algorithm needs to observe its environment, learn by testing new actions (some of which may not be immediately optimal), and reuse its previous experiences. The learning subsequently occurs through millions of trials and errors. Academics and practitioners are applying RL in a similar way in investment strategies where the agent could be a virtual trader who follows certain trading rules (the actions) in a specific market (the environment) to maximize its profits (its reward). The success of RL in dealing with the complexities of financial markets is still an open question.

EXAMPLE 8

Deep Neural Networks

Glen Mitsui is the chief investment officer for a large Australian state's Public Employees' Pension Fund (PEPF), which currently has assets under management (AUM) of A$20 billion. The fund manages one-quarter of its assets internally, with A$5 billion mostly in domestic government and corporate fixed-income instruments and domestic equities. The remaining three-quarters of AUM, or A$15 billion, is managed by nearly 100 mostly active external asset managers and is invested in a wide range of asset classes, including foreign fixed income and equities, domestic and foreign hedge funds, REITs, commodities, and derivatives.

PEPF has a small staff of four investment professionals tasked with selecting and monitoring these external managers to whom it pays more than A$400 million in fees annually. Performance (compared to appropriate benchmarks) of many of PEPF's external managers has been lagging over the past several years. After studying the situation, Mitsui concludes that style drift may be an important factor in explaining such underperformance, for which PEPF is not happy to pay. Mitsui believes that machine learning may help and consults with Frank Monroe, professor of data analysis at Epsilon University.

Monroe suggests using a deep neural network model that collects and analyzes the real-time trading data of PEPF's external managers and compares them to well-known investment styles (e.g., high dividend, minimum volatility, momentum, growth, value) to detect potential style drift. Mitsui arranges for Monroe to meet with PEPF's investment committee (IC) to discuss the matter. As a junior data analyst working with Monroe, you must help him satisfy the following requests from the IC:

1. Define a deep neural network.

Solution:

A deep neural network is a neural network (NN) with many hidden layers (at least 2 but often more than 20). NNs and DNNs have been successfully applied to a wide variety of complex tasks characterized by non-linearities and interactions among features, particularly pattern recognition problems.

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2. Evaluate Monroe's opinion on the applicability of deep neural networks to Mitsui's problem.

Solution:

Mitsui wants to detect patterns of potential style drift in the daily trading of nearly 100 external asset managers in many markets. This task will involve the processing of huge amounts of complicated data. Monroe is correct that a DNN is well suited to PEPF's needs.

3. Describe the functions of the three groups of layers of a deep neural network.

Solution:

The input layer, the hidden layers, and the output layer constitute the three groups of layers of DNNs. The input layer receives the inputs (i.e., features) and has as many nodes as there are dimensions of the feature set. The hidden layers consist of nodes, each comprising a summation operator and an activation function that are connected by links. These hidden layers are, in effect, where the model is learned. The final layer, the output layer, produces a set of probabilities of an observation being in any of the target style categories (each represented by a node in the output layer). For example, if there are three target style categories, then three nodes in the output layer are activated to produce outputs that sum to one. So, output (Style Category I, 0.7; Style Category II, 0.2; Style Category III, 0.1) would indicate that the model assigns the greatest probability to an observation being in Style Category I and the least probability to Style Category III. The DNN assigns the observation to the style category with the highest probability.

17 CASE STUDY: DEEP NEURAL NETWORK–BASED EQUITY FACTOR MODEL

- ☐ describe neural networks, deep learning nets, and reinforcement learning

The following case study was developed and written by Matthew Dixon, PhD, FRM.

An investment manager wants to select stocks based on their predicted performance using a fundamental equity factor model. She seeks to capture superior performance from stocks with the largest excess return using a non-linear factor model and so chooses a deep neural network to predict the stock returns. The goal of this mini-case study is to demonstrate the application of deep neural networks to fundamental equity factor modeling. We shall focus on using feed-forward (i.e., forward propagation) network regression in place of ordinary least squares linear regression. Since neural networks are prone to over-fitting, we shall use LASSO penalization, the same penalty score–based approach used previously with regression, to mitigate this issue.

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Introduction

Cross-sectional fundamental factor models are used extensively by investment managers to capture the effects of company-specific factors on individual securities. A fixed universe of N assets is first chosen, together with a set of K fundamental factors. Each asset's sensitivity (i.e., exposure or loading) to a fundamental factor is represented by beta, B, and the factors are represented by factor returns (f_t). There are two standard approaches to estimating a factor model: (i) adopt time-series regression (TSR) to recover loadings if factors are known or (ii) use cross-sectional regression (CSR) to recover factor returns from known loadings. We shall follow the CSR approach; the factor exposures are used to predict a stock's return (r_t) by estimating the factor returns using multivariate linear regression (where ε_t is the model error at time t):

$$r_t = Bf_t + \varepsilon_t.$$

However, this CSR model is too simplistic to capture non-linear relationships between stock returns and fundamental factors. So, instead we use a deep neural network to learn the non-linear relationships between the betas (B) and asset returns (r_t) at each time t. The goal of deep learning is to find the network weights which minimize the out-of-sample mean squared error (MSE) between the predicted stock returns, $\hat{r}$, and the observed stock returns, r. We shall see that simply increasing the number of neurons in the network will increase predictive performance using the in-sample data but to the detriment of out-of-sample performance; this phenomenon is the bias–variance trade-off. To mitigate this effect, we add a LASSO penalty term to the loss function to automatically shrink the number of non-zero weights in the network. In doing so, we shall see that this leads to better out-of-sample predictive performance.

Note that each weight corresponds to a link between a node in the previous and current layer. Reducing the number of weights generally means that the number of connections—not the number of nodes—is reduced. The exception is when all weights from the neurons in the previous layer are set to zero—in which case the number of nodes in the current layer would be reduced. In the special case when the previous layer is the input layer, the number of features is also reduced.

We shall illustrate the data preparation and the neural network fitting using six fundamental equity factors. This choice of number and type of fundamental factor is arbitrary, and an investment manager may use many more factors in her or his model, often representing industry sectors and sub-sectors using dummy variables.

Data Description

A description of the stock price and fundamental equity factor data used for training and evaluating the neural network is shown in Exhibit 31.

Exhibit 31: Dataset of S&P 500 Stocks and Fundamental Factors

Description:

A subset of S&P 500 Index stocks, historical monthly adjusted closing prices, and corresponding monthly fundamental factor loadings.

Time period: June 2010 to November 2018

Number of periods: 101

Number of stocks (N): 218 stocks

Number of features (K): 6

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Features: Fundamental equity factors:

1. Current enterprise value (i.e., market values of equity + preferred stock + debt − cash − short-term investments)
2. Current enterprise value to trailing 12-month EBITDA
3. Price-to-sales ratio
4. Price-to-earnings ratio
5. Price-to-book ratio
6. Log of stock's market capitalization (i.e., share price × number of shares outstanding)

Output: Monthly return for each stock over the following month.

We define the universe as the top 250 stocks from the S&P 500, ranked by market capitalization as of June 2010. All stock prices and factor loadings are sourced from Bloomberg. An illustrative extract of the data is given in Exhibit 32. Note that after removing stocks with missing factor loadings, we are left with 218 stocks.

Exhibit 32: Extract of Six Factor Loadings and Return for Three Selected Stocks

TICKER	CURR_EV ($ Mil.)	CURR_EV_TO_ T12M_EBITDA (X)	PX_TO_ SALES (X)	PX_TO_ EARN (X)	PX_TO_ BOOK (X)	LOG_CAP ($ Mil.)	RETURN (%)
SWK	10,775.676	30.328	1.138	16.985	1.346	9.082970	−0.132996
STZ	7,433.553	15.653	1.052	10.324	1.480	8.142253	−0.133333
SRE	19,587.124	10.497	1.286	10.597	1.223	9.314892	−0.109589

Experimental Design

The method used to train the deep neural network is time-series cross-validation (i.e., walk-forward optimization), as depicted in Exhibit 33. At each time period, the investment manager fits a new model; each factor (f_1 to f_6) is a feature in the network, and the loadings of the factors for each stock is a feature vector observation (i.e., the set of observations for each stock for each period), leading to $N = 218$ observations of pairs of feature vectors and output (monthly return, r_t) in the training set per period. The network is initially trained at period t, and then it is tested over the next period, $t + 1$, which also has $N = 218$ observations of pairs of feature vectors and output. In the next iteration, the $t + 1$ data become the new training set and the revised model is tested on the $t + 2$ data. The walk-forward optimization of the neural network continues until the last iteration: model training with $t + 99$ data (from Period 100) and testing with $t + 100$ data (from the last period, 101).

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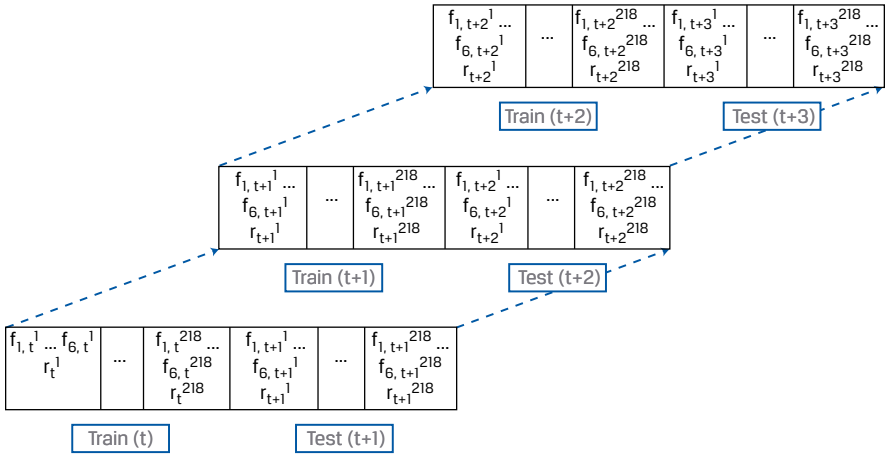
Exhibit 33: Time-Series Cross-Validation on Asset Returns (Walk-Forward Optimization)—The First Three Iterations



We use a feed-forward neural network with six input nodes (i.e., neurons), two hidden layers, and one output neuron. There are 50 neurons in each hidden layer to intentionally over-specify the number of parameters needed in the model, meaning bias (variance) is substantially lower (higher) than optimal. LASSO penalization is then used to automatically shrink the parameter set. Additionally, it is important for the number of nodes in each hidden layer not to exceed the number of observations in the training set (50 nodes per layer versus 218 observations). The model training in period t involves finding the optimal bias-versus-variance trade-off. Once fitted, we record the in-sample MSE and the out-of-sample MSE in addition to the optimal regularization parameter. This procedure is then repeated sequentially over the horizon of 100 remaining periods, tuning the hyperparameters at each stage using cross-validation. The end result of this procedure is a fitted model, trained monthly on the current cross-sectional data and for which hyperparameters have been tuned at each step.

Results

Exhibit 34 presents the results from model evaluation; it compares the in-sample and out-of-sample MSEs of the deep neural network over all 101 months. Note that the out-of-sample error (dotted line) is typically significantly larger than the in-sample error (solid line). However, as the time periods pass and the model is repeatedly trained and tested, the difference between the out-of-sample and in-sample MSEs narrows dramatically.

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Exhibit 34: In-Sample and Out-of-Sample MSE for Each Training and Testing Period

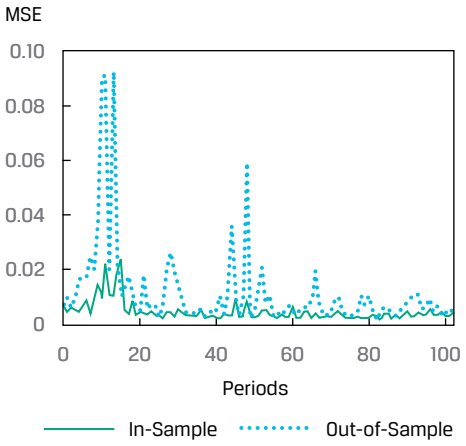

Exhibit 35 shows the effect of LASSO regularization on the in-sample MSE (lower panel, B) and the out-of-sample MSE (upper panel, A) for the first iteration of the time-series cross-validation (training with data from period t and testing with data from period t +1). The degree of LASSO regularization needed is found by cross-validation using 50 neurons in each hidden layer. Increasing the LASSO regularization, which reduces the number of non-zero weights in the model, introduces more bias and hence increases the in-sample error. Conversely, increasing the LASSO regularization reduces the model's variance and thereby reduces the out-of-sample error. Overall, the amount of LASSO regularization needed is significant, at 0.10; typically the regularization hyperparameter is between 0.001 and 1.0. Also, the out-of-sample and in-sample MSEs have not yet converged. There is still a substantial gap, of roughly 0.0051 (= 0.01025 – 0.0052), and the slope of the curves in each plot suggests the optimal value of the regularization hyperparameter is significantly more than 0.10. Note that the value of the regularization hyperparameter is not interpretable and does not correspond to the number of weights eliminated. Suffice it to say, the larger the value of the regularization hyperparameter, the more the loss is being penalized.

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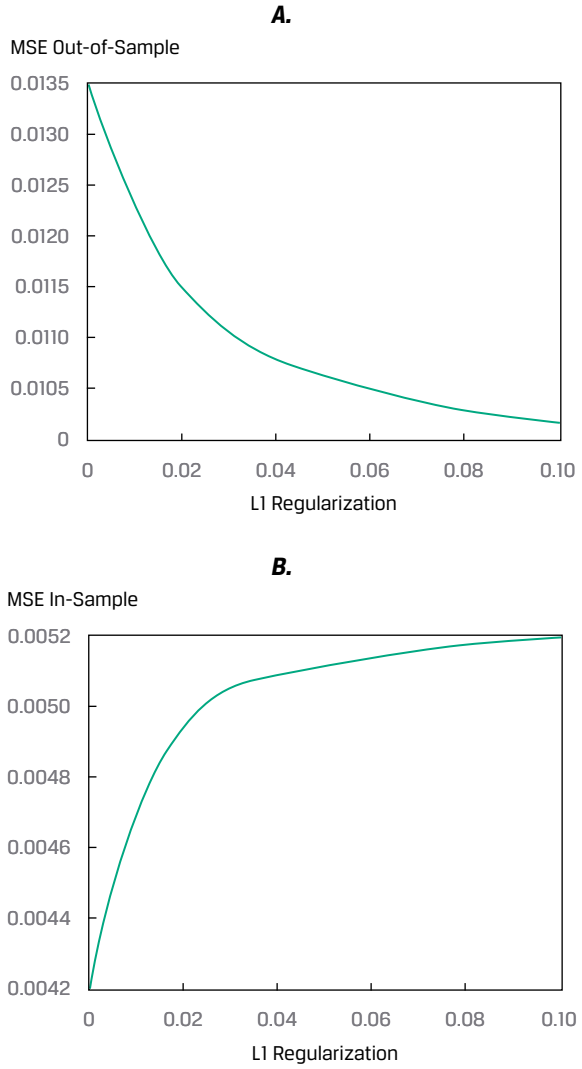
Exhibit 35: LASSO Regularization for Optimizing Bias–Variance Trade-Off (First Iteration)



It is important to recognize that although the out-of-sample MSE of this deep learning neural network is key to characterizing its predictive performance, it does not necessarily follow that a stock selection strategy based on the neural network will be successful. This is because the neural network predicts the next month's expected (i.e., mean) asset returns and not the full distribution of returns. Hence a simple stock selection strategy—measured by information ratios (recall the information ratio, or IR, is alpha divided by nonsystematic risk, so it measures the abnormal return per unit of risk for a well-diversified portfolio) of the portfolio returns—that selects stocks ranked by predicted returns will not necessarily lead to positive information ratios.

Exhibit 36 presents the information ratios found by back-testing a simple stock selection strategy that picks the top performing stocks determined by the neural network's forecasted returns realized in month t +1 using features observed in month t. Note these IRs do not account for transaction costs, interest rates, or any other fees. The upper panel (A) shows the best-case scenario; the neural network in-sample prediction is used to select the n (where n is 10, 15, 20, or 25) top performing stocks. The IRs are shown for each of the different-sized portfolios; they range from 0.697

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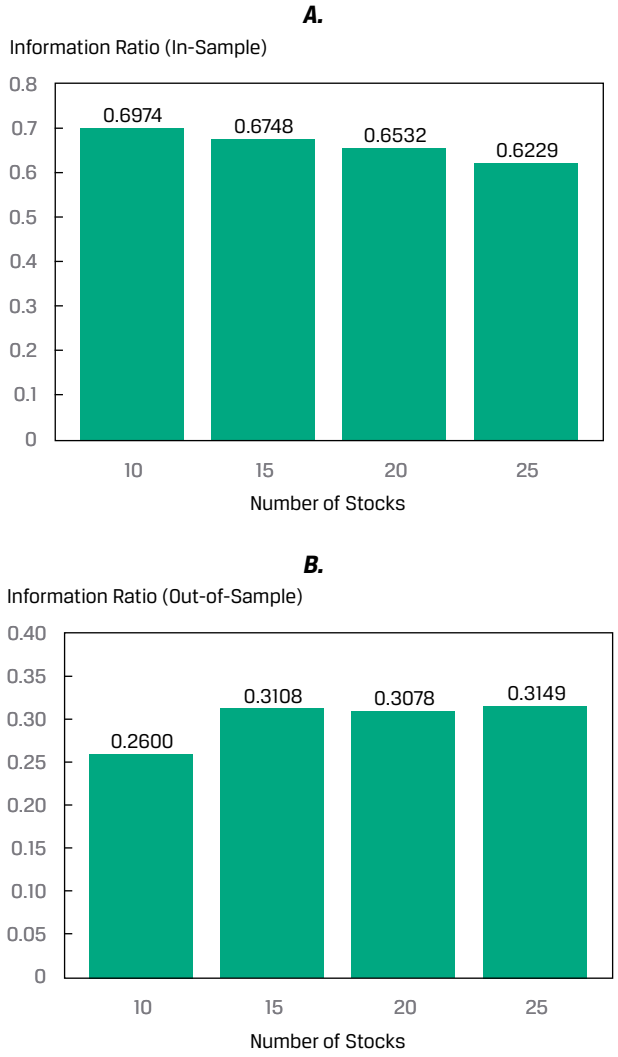
to 0.623. Note that as a rule of thumb, IRs in the range of 0.40–0.60 are considered quite good. The lower panel (B) shows the IRs from back-test results for the same strategy applied to the out-of-sample data. The out-of-sample IRs range from 0.260 to 0.315 and so are substantially smaller than in-sample IRs.

Exhibit 36: Information Ratios from Back-Testing a Stock Selection Strategy Using Top Performers from the Neural Network



Importantly, the out-of-sample performance provides the most realistic assessment of the likely future investment performance from applying this deep learning neural network to stock selection. It is a baseline for further model refinements, including adding more fundamental and macroeconomic factors. With such refinements, it can be expected that the out-of-sample IRs should improve substantially.

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EXAMPLE 9

Deep Learning–Based Fundamental Factor Model

A research analyst, Jane Hinton, has been tasked with further developing the deep learning–based fundamental factor model. She decides to refine the model by adding four more fundamental factors (such as debt leverage and R&D intensity) given by firm characteristics and by including dummy variables for 11 industrial sectors. Moreover, she additionally expands the universe of stocks to 420 from 218 by using a supplementary data source.

1. Describe how Jane would modify the inputs of the neural network architecture for this new dataset.

Solution:

Jane adds four more fundamental factors and 11 dummy variables, to represent each industrial sector, for a total of 21 (= 4 + 11 + 6) features. Therefore, the refined neural network will have 21 input neurons. The output layer will remain the same. Note that concerns of collinearity of the features through the dummy variables or high correlation, which are problematic for linear regression, are not an issue for a deep learning–based model.

2. Describe the size of the new training and test datasets.

Solution:

There are now data on 420 stocks, for each of the 101 time periods, consisting of factor loadings for the 21 features and the monthly return for each stock. Per the time-series cross-validation method, the test dataset in the current iteration will become the training dataset in the next iteration.

3. Describe any additional changes to the architecture and hyperparameters of the neural network that Jane would likely need to make to ensure good performance of the network.

Solution:

Jane should find the new optimal LASSO regularization hyperparameter using time-series cross-validation. Alternatively, she may find the optimal bias–variance trade-off by first increasing the number of neurons in the hidden layers and then performing the cross-validation.

4. Explain how Jane should evaluate whether the new model leads to improved portfolio performance.

Solution:

Once Jane has found the optimal LASSO hyperparameter and network architecture, she will use the model to forecast the out-of-sample monthly asset returns (i.e., the model forecasts from factor loadings which are not in the training set). She will then rank and select the top predicted performers and finally measure the realized monthly portfolio return. She will then repeat the experiment by moving forward one month in the dataset and repeating the out-of-sample forecast of the asset returns, until she has generated forecasts for all time periods. Finally, Jane will calculate the information ratios from the mean and standard deviation of the monthly portfolio excess returns.

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EXAMPLE 10

Summing Up the Major Types of Machine Learning

1. As used in supervised machine learning, classification problems involve the following *except*:

   A. binary target variables.

   B. continuous target variables.

   C. categorical target variables.

Solution:

B is correct. A and C are incorrect because when the target variable is binary or categorical (not continuous), the problem is a classification problem.

2. Which of the following *best* describes penalized regression? Penalized regression:

   A. is unrelated to multiple linear regression.

   B. involves a penalty term that is added to the predicted target variable.

   C. is a category of general linear models used when the number of features and overfitting are concerns.

Solution:

C is correct. A is incorrect because penalized regression is related to multiple linear regression. B is incorrect because penalized regression involves adding a penalty term to the sum of the squared regression residuals.

3. CART is *best* described as:

   A. an unsupervised ML algorithm.

   B. a clustering algorithm based on decision trees.

   C. a supervised ML algorithm that accounts for non-linear relationships among the features.

Solution:

C is correct. A is incorrect because CART is a supervised ML algorithm. B is incorrect because CART is a classification and regression algorithm, not a clustering algorithm.

4. A neural network is *best* described as a technique for machine learning that is:

   A. exactly modeled on the human nervous system.

   B. based on layers of nodes connected by links when the relationships among the features are usually non-linear.

   C. based on a tree structure of nodes when the relationships among the features are linear.

Solution:

B is correct. A is incorrect because neural networks are not exactly modeled on the human nervous system. C is incorrect because neural networks are not based on a tree structure of nodes when the relationships among the features are linear.

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5. Hierarchical clustering is *best* described as a technique in which:

   A. the grouping of observations is unsupervised.

   B. features are grouped into a pre-specified number, *k*, of clusters.

   C. observations are classified according to predetermined labels.

Solution:

A is correct. B is incorrect because it refers to *k*-means clustering. C is incorrect because it refers to classification, which involves supervised learning.

6. Dimension reduction techniques are *best* described as a means to reduce a set of features to a manageable size:

   A. without regard for the variation in the data.

   B. while increasing the variation in the data.

   C. while retaining as much of the variation in the data as possible.

Solution:

C is correct because dimension reduction techniques, such as PCA, are aimed at reducing the feature set to a manageable size while retaining as much of the variation in the data as possible.

CHOOSING AN APPROPRIATE ML ALGORITHM

18

- ☐ describe supervised machine learning algorithms—including penalized regression, support vector machine, k-nearest neighbor, classification and regression tree, ensemble learning, and random forest—and determine the problems for which they are best suited
- ☐ describe unsupervised machine learning algorithms—including principal components analysis, k-means clustering, and hierarchical clustering—and determine the problems for which they are best suited

Exhibit 37 presents a simplified decision flowchart for choosing among the machine learning algorithms which have been discussed. The dark-shaded ovals contain the supervised ML algorithms, the light-shaded ovals contain the unsupervised ML algorithms, and the key questions to consider are shown in the unshaded rounded rectangles.

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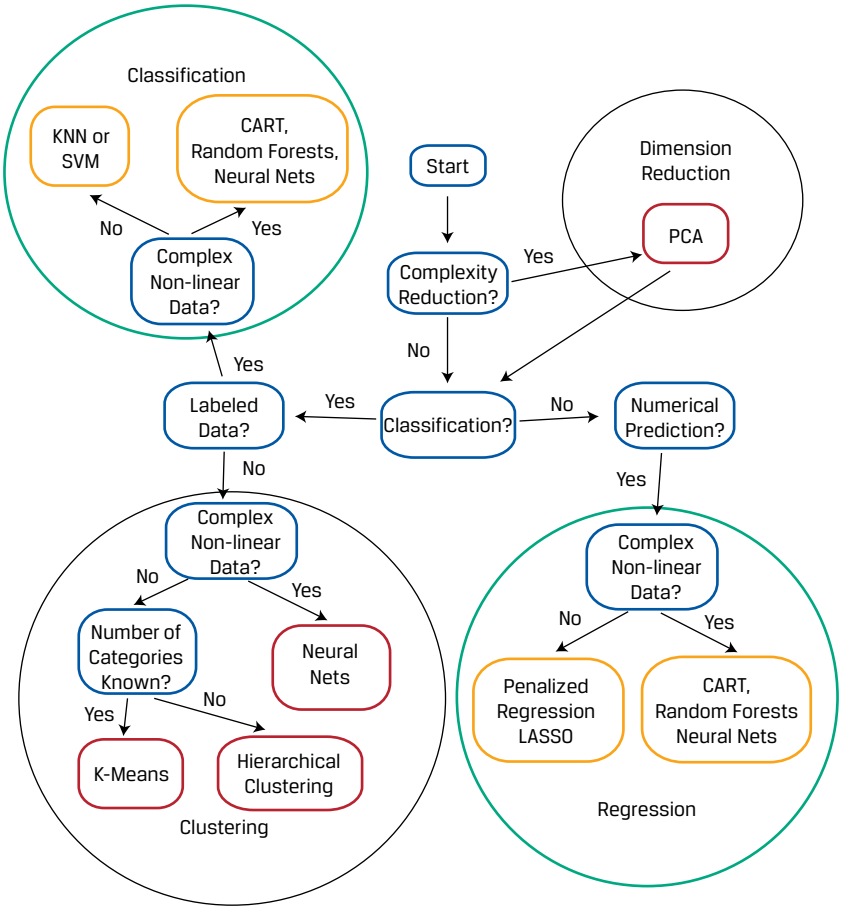
Exhibit 37: Stylized Decision Flowchart for Choosing ML Algorithms



First, start by asking, Are the data complex, having many features that are highly correlated? If yes, then dimension reduction using principal components analysis is appropriate.

Next, is the problem one of classification or numerical prediction? If numerical prediction, then depending on whether the data have non-linear characteristics, the choice of ML algorithms is from a set of regression algorithms—either penalized regression/LASSO for linear data or CART, random forest, or neural networks for non-linear data.

If the problem is one of classification, then depending on whether the data are labeled, the choice is either from a set of classification algorithms using labeled data or from a set of clustering algorithms using unlabeled data.

If the data are labeled, then depending on whether the data have non-linear characteristics, the choice of classification algorithm would be *K*-nearest neighbor and support vector machine for linear data or CART, random forest, or neural networks (or deep neural networks) for non-linear data.

Finally, if the data are unlabeled, the choice of clustering algorithm depends on whether the data have non-linear characteristics. The choice of clustering algorithm would be neural networks (or deep neural networks) for non-linear data or for linear data, *K*-means with a known number of categories and hierarchical clustering with an unknown number of categories.

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SUMMARY

Machine learning methods are gaining usage at many stages in the investment management value chain. Among the major points made are the following:

- Machine learning aims at extracting knowledge from large amounts of data by learning from known examples to determine an underlying structure in the data. The emphasis is on generating structure or predictions without human intervention. An elementary way to think of ML algorithms is to "find the pattern, apply the pattern."
- Supervised learning depends on having labeled training data as well as matched sets of observed inputs (X's, or features) and the associated output (Y, or target). Supervised learning can be divided into two categories: regression and classification. If the target variable to be predicted is continuous, then the task is one of regression. If the target variable is categorical or ordinal (e.g., determining a firm's rating), then it is a classification problem.
- With unsupervised learning, algorithms are trained with no labeled data, so they must infer relations between features, summarize them, or present underlying structure in their distributions that has not been explicitly provided. Two important types of problems well suited to unsupervised ML are dimension reduction and clustering.
- In deep learning, sophisticated algorithms address complex tasks (e.g., image classification, natural language processing). Deep learning is based on neural networks, highly flexible ML algorithms for solving a variety of supervised and unsupervised tasks characterized by large datasets, non-linearities, and interactions among features. In reinforcement learning, a computer learns from interacting with itself or data generated by the same algorithm.
- Generalization describes the degree to which an ML model retains its explanatory power when predicting out-of-sample. Overfitting, a primary reason for lack of generalization, is the tendency of ML algorithms to tailor models to the training data at the expense of generalization to new data points.
- Bias error is the degree to which a model fits the training data. Variance error describes how much a model's results change in response to new data from validation and test samples. Base error is due to randomness in the data. Out-of-sample error equals bias error plus variance error plus base error.
- K-fold cross-validation is a technique for mitigating the holdout sample problem (excessive reduction of the training set size). The data (excluding test sample and fresh data) are shuffled randomly and then divided into k equal sub-samples, with $k - 1$ samples used as training samples and one sample, the kth, used as a validation sample.
- Regularization describes methods that reduce statistical variability in high-dimensional data estimation or prediction problems via reducing model complexity.
- LASSO (least absolute shrinkage and selection operator) is a popular type of penalized regression where the penalty term involves summing the absolute values of the regression coefficients. The greater the number of included features, the larger the penalty. So, a feature must make a sufficient contribution to model fit to offset the penalty from including it.

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- Support vector machine (SVM) is a classifier that aims to seek the optimal hyperplane—the one that separates the two sets of data points by the maximum margin (and thus is typically used for classification).
- *K*-nearest neighbor (KNN) is a supervised learning technique most often used for classification. The idea is to classify a new observation by finding similarities ("nearness") between it and its *k*-nearest neighbors in the existing dataset.
- Classification and regression tree (CART) can be applied to predict either a categorical target variable, producing a classification tree, or a continuous target variable, producing a regression tree.
- A binary CART is a combination of an initial root node, decision nodes, and terminal nodes. The root node and each decision node represent a single feature (*f*) and a cutoff value (*c*) for that feature. The CART algorithm iteratively partitions the data into sub-groups until terminal nodes are formed that contain the predicted label.
- Ensemble learning is a technique of combining the predictions from a collection of models. It typically produces more accurate and more stable predictions than any single model.
- A random forest classifier is a collection of many different decision trees generated by a bagging method or by randomly reducing the number of features available during training.
- Principal components analysis (PCA) is an unsupervised ML algorithm that reduces highly correlated features into fewer uncorrelated composite variables by transforming the feature covariance matrix. PCA produces eigenvectors that define the principal components (i.e., the new uncorrelated composite variables) and eigenvalues, which give the proportion of total variance in the initial data that is explained by each eigenvector and its associated principal component.
- *K*-means is an unsupervised ML algorithm that partitions observations into a fixed number (*k*) of non-overlapping clusters. Each cluster is characterized by its centroid, and each observation belongs to the cluster with the centroid to which that observation is closest.
- Hierarchical clustering is an unsupervised iterative algorithm that is used to build a hierarchy of clusters. Two main strategies are used to define the intermediary clusters (i.e., those clusters between the initial dataset and the final set of clustered data).
- Agglomerative (bottom-up) hierarchical clustering begins with each observation being its own cluster. Then, the algorithm finds the two closest clusters, defined by some measure of distance, and combines them into a new, larger cluster. This process is repeated until all observations are clumped into a single cluster.
- Divisive (top-down) hierarchical clustering starts with all observations belonging to a single cluster. The observations are then divided into two clusters based on some measure of distance. The algorithm then progressively partitions the intermediate clusters into smaller clusters until each cluster contains only one observation.
- Neural networks consist of nodes connected by links. They have three types of layers: an input layer, hidden layers, and an output layer. Learning takes place in the hidden layer nodes, each of which consists of a summation

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operator and an activation function. Neural networks have been successfully applied to a variety of investment tasks characterized by non-linearities and complex interactions among variables.

- Neural networks with many hidden layers (at least 2 but often more than 20) are known as deep neural networks (DNNs) and are the backbone of the artificial intelligence revolution.
- Reinforcement learning (RL) involves an agent that should perform actions that will maximize its rewards over time, taking into consideration the constraints of its environment.

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PRACTICE PROBLEMS

The following information relates to questions 1-10

Alef Associates manages a long-only fund specializing in global smallcap equities. Since its founding a decade ago, Alef maintains a portfolio of 100 stocks (out of an eligible universe of about 10,000 stocks). Some of these holdings are the result of screening the universe for attractive stocks based on several ratios that use readily available market and accounting data; others are the result of investment ideas generated by Alef's professional staff of five securities analysts and two portfolio managers.

Although Alef's investment performance has been good, its Chief Investment Officer, Paul Moresanu, is contemplating a change in the investment process aimed at achieving even better returns. After attending multiple workshops and being approached by data vendors, Moresanu feels that data science should play a role in the way Alef selects its investments. He has also noticed that much of Alef's past outperformance is due to stocks that became takeover targets. After some research and reflection, Moresanu writes the following email to the Alef's CEO.

Exhibit 1:

Subject: Investment Process Reorganization

I have been thinking about modernizing the way we select stock investments. Given that our past success has put Alef Associates in an excellent financial position, now seems to be a good time to invest in our future. What I propose is that we continue managing a portfolio of 100 global small-cap stocks but restructure our process to benefit from machine learning (ML). Importantly, the new process will still allow a role for human insight, for example, in providing domain knowledge. In addition, I think we should make a special effort to identify companies that are likely to be acquired. Specifically, I suggest following the four steps which would be repeated every quarter.

Step 1 We apply ML techniques to a model including fundamental and technical variables (features) to predict next quarter's return for each of the 100 stocks currently in our portfolio. Then, the 20 stocks with the lowest estimated return are identified for replacement.

Step 2 We utilize ML techniques to divide our investable universe of about 10,000 stocks into 20 different groups, based on a wide variety of the most relevant financial and non-financial characteristics. The idea is to prevent unintended portfolio concentration by selecting stocks from each of these distinct groups.

Step 3 For each of the 20 different groups, we use labeled data to train a model that will predict the five stocks (in any given group) that are most likely to become acquisition targets in the next one year.

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Step 4 Our five experienced securities analysts are each assigned four of the groups, and then each analyst selects their one best stock pick from each of their assigned groups. These 20 "high-conviction" stocks will be added to our portfolio (in replacement of the 20 relatively underperforming stocks to be sold in Step 1).

A couple of additional comments related to the above:

Comment 1 The ML algorithms will require large amounts of data. We would first need to explore using free or inexpensive historical datasets and then evaluate their usefulness for the ML-based stock selection processes before deciding on using data that requires subscription.

Comment 2 As time passes, we expect to find additional ways to apply ML techniques to refine Alef's investment processes.

What do you think?

Paul Moresanu

---

1. The machine learning techniques appropriate for executing Step 1 are *most* likely to be based on:

   A. regression

   B. classification

   C. clustering

2. Assuming regularization is utilized in the machine learning technique used for executing Step 1, which of the following ML models would be *least* appropriate:

   A. Regression tree with pruning.

   B. LASSO with lambda (λ) equal to 0.

   C. LASSO with lambda (λ) between 0.5 and 1.

3. Which of the following machine learning techniques is *most* appropriate for executing Step 2:

   A. K-Means Clustering

   B. Principal Components Analysis (PCA)

   C. Classification and Regression Trees (CART)

4. The hyperparameter in the ML model to be used for accomplishing Step 2 is?

   A. 100, the number of small-cap stocks in Alef's portfolio.

   B. 10,000, the eligible universe of small-cap stocks in which Alef can potentially invest.

   C. 20, the number of different groups (i.e. clusters) into which the eligible universe of small-cap stocks will be divided.

5. The target variable for the labelled training data to be used in Step 3 is *most* likely which one of the following?

   A. A continuous target variable.

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B. A categorical target variable.

C. An ordinal target variable.

6. Comparing two ML models that could be used to accomplish Step 3, which statement(s) *best* describe(s) the advantages of using Classification and Regression Trees (CART) instead of K-Nearest Neighbor (KNN)?

Statement 1	For CART there is no requirement to specify an initial hyperparameter (like K).
Statement 2	For CART there is no requirement to specify a similarity (or distance) measure.
Statement 3	For CART the output provides a visual explanation for the prediction.

A. Statement 1 only.

B. Statement 3 only.

C. Statements 1, 2, and 3.

7. Assuming a Classification and Regression Tree (CART) model is used to accomplish Step 3, which of the following is *most* likely to result in model overfitting?

A. Using the k-fold cross validation method

B. Including an overfitting penalty (i.e., regularization term).

C. Using a fitting curve to select a model with low bias error and high variance error.

8. Assuming a Classification and Regression Tree (CART) model is initially used to accomplish Step 3, as a further step which of the following techniques is most likely to result in more accurate predictions?

A. Discarding CART and using the predictions of a Support Vector Machine (SVM) model instead.

B. Discarding CART and using the predictions of a K-Nearest Neighbor (KNN) model instead.

C. Combining the predictions of the CART model with the predictions of other models – such as logistic regression, SVM, and KNN – via ensemble learning.

9. Regarding Comment #2, Moresanu has been thinking about the applications of neural networks (NNs) and deep learning (DL) to investment management. Which statement(s) *best* describe(s) the tasks for which NNs and DL are well-suited?

Statement 1	NNs and DL are well-suited for image and speech recognition, and natural language processing.
Statement 2	NNs and DL are well-suited for developing single variable ordinary least squares regression models.

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Statement 3 NNs and DL are well-suited for modelling non-linearities and complex interactions among many features.

A. Statement 2 only.

B. Statements 1 and 3.

C. Statements 1, 2 and 3.

10. Regarding neural networks (NNs) that Alef might potentially implement, which of the following statements is *least* accurate?

A. NNs must have at least 10 hidden layers to be considered deep learning nets.

B. The activation function in a node operates like a light dimmer switch since it decreases or increases the strength of the total net input.

C. The summation operator receives input values, multiplies each by a weight, sums up the weighted values into the total net input, and passes it to the activation function.

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SOLUTIONS

1. A is correct. The target variable (quarterly return) is continuous, hence this calls for a supervised machine learning based regression model.

   B is incorrect, since classification uses categorical or ordinal target variables, while in Step 1 the target variable (quarterly return) is continuous.

   C is incorrect, since clustering involves unsupervised machine learning so does not have a target variable.

2. B is correct. It is least appropriate because with LASSO, when $\lambda = 0$ the penalty (i.e., regularization) term reduces to zero, so there is no regularization and the regression is equivalent to an ordinary least squares (OLS) regression.

   A is incorrect. With Classification and Regression Trees (CART), one way that regularization can be implemented is via pruning which will reduce the size of the regression tree—sections that provide little explanatory power are pruned (i.e., removed).

   C is incorrect. With LASSO, when λ is between 0.5 and 1 the relatively large penalty (i.e., regularization) term requires that a feature makes a sufficient contribution to model fit to offset the penalty from including it in the model.

3. A is correct. K-Means clustering is an unsupervised machine learning algorithm which repeatedly partitions observations into a fixed number, *k*, of non-overlapping clusters (i.e., groups).

   B is incorrect. Principal Components Analysis is a long-established statistical method for dimension reduction, not clustering. PCA aims to summarize or reduce highly correlated features of data into a few main, uncorrelated composite variables.

   C is incorrect. CART is a supervised machine learning technique that is most commonly applied to binary classification or regression.

4. C is correct. Here, 20 is a hyperparameter (in the K-Means algorithm), which is a parameter whose value must be set by the researcher before learning begins.

   A is incorrect, because it is not a hyperparameter. It is just the size (number of stocks) of Alef's portfolio.

   B is incorrect, because it is not a hyperparameter. It is just the size (number of stocks) of Alef's eligible universe.

5. B is correct. To predict which stocks are likely to become acquisition targets, the ML model would need to be trained on categorical labelled data having the following two categories: "0" for "not acquisition target", and "1" for "acquisition target".

   A is incorrect, because the target variable is categorical, not continuous.

   C is incorrect, because the target variable is categorical, not ordinal (i.e., 1st, 2nd, 3rd, etc.).

6. C is correct. The advantages of using CART over KNN to classify companies into two categories ("not acquisition target" and "acquisition target"), include all of the following: For CART there are no requirements to specify an initial hyperparameter (like K) or a similarity (or distance) measure as with KNN, and CART provides a visual explanation for the prediction (i.e., the feature variables and their cut-off values at each node).

   A is incorrect, because CART provides all of the advantages indicated in Statements 1, 2 and 3.

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B is incorrect, because CART provides all of the advantages indicated in Statements 1, 2 and 3.

7. C is correct. A fitting curve shows the trade-off between bias error and variance error for various potential models. A model with low bias error and high variance error is, by definition, overfitted.

   A is incorrect, because there are two common methods to reduce overfitting, one of which is proper data sampling and cross-validation. K-fold cross validation is such a method for estimating out-of-sample error directly by determining the errorin validation samples.

   B is incorrect, because there are two common methods to reduce overfitting, one of which is preventing the algorithm from getting too complex during selection and training, which requires estimating an overfitting penalty.

8. C is correct. Ensemble learning is the technique of combining the predictions from a collection of models, and it typically produces more accurate and more stable predictions than the best single model.

   A is incorrect, because a single model will have a certain error rate and will make noisy predictions. By taking the average result of many predictions from many models (i.e., ensemble learning) one can expect to achieve a reduction in noise as the average result converges towards a more accurate prediction.

   B is incorrect, because a single model will have a certain error rate and will make noisy predictions. By taking the average result of many predictions from many models (i.e., ensemble learning) one can expect to achieve a reduction in noise as the average result converges towards a more accurate prediction.

9. B is correct. NNs and DL are well-suited for addressing highly complex machine learning tasks, such as image classification, face recognition, speech recognition and natural language processing. These complicated tasks are characterized by non-linearities and complex interactions between large numbers of feature inputs.

   A is incorrect, because NNs and DL are well-suited for addressing highly complex machine learning tasks, not simple single variable OLS regression models.

   C is incorrect, because NNs and DL are well-suited for addressing highly complex machine learning tasks, not simple single variable OLS regression models.

10. A is correct. It is the least accurate answer because neural networks with many hidden layers—at least 2, but often more than 20 hidden layers—are known as deep learning nets.

    B is incorrect, because the node's activation function operates like a light dimmer switch which decreases or increases the strength of the (total net) input.

    C is incorrect, because the node's summation operator multiplies each (input) value by a weight and sums up the weighted values to form the total net input. The total net input is then passed to the activation function.

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LEARNING MODULE

7

Big Data Projects

by Sreekanth Mallikarjun, PhD, and Ahmed Abbasi, PhD.

Sreekanth Mallikarjun, PhD, is at Reorg (USA) and the University of Virginia, School of Data Science (USA). Ahmed Abbasi, PhD, is at the University of Virginia, McIntire School of Commerce (USA).

LEARNING OUTCOMES

Mastery	*The candidate should be able to:*
☐	identify and explain steps in a data analysis project
☐	describe objectives, steps, and examples of preparing and wrangling data
☐	evaluate the fit of a machine learning algorithm
☐	describe objectives, methods, and examples of data exploration
☐	describe methods for extracting, selecting and engineering features from textual data
☐	describe objectives, steps, and techniques in model training
☐	describe preparing, wrangling, and exploring text-based data for financial forecasting

INTRODUCTION 1

Big data (also referred to as alternative data) encompasses data generated by financial markets (e.g., stock and bond prices), businesses (e.g., company financials, production volumes), governments (e.g., economic and trade data), individuals (e.g., credit card purchases, social media posts), sensors (e.g., satellite imagery, traffic patterns), and the Internet of Things, or IoT, (i.e., the network of interrelated digital devices that can transfer data among themselves without human interaction). A veritable explosion in big data has occurred over the past decade or so, especially in unstructured data generated from social media (e.g., posts, tweets, blogs), email and text communications, web traffic, online news sites, electronic images, and other electronic information sources. The prospects are for exponential growth in big data to continue.

Investment managers are increasingly using big data in their investment processes as they strive to discover signals embedded in such data that can provide them with an information edge. They seek to augment structured data with a plethora of unstructured data to develop improved forecasts of trends in asset prices, detect anomalies,

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etc. A typical example involves a fund manager using financial text data from 10-K reports for forecasting stock sentiment (i.e., positive or negative), which can then be used as an input to a more comprehensive forecasting model that includes corporate financial data.

Unlike structured data (numbers and values) that can be readily organized into data tables to be read and analyzed by computers, unstructured data typically require specific methods of preparation and refinement before being usable by machines (i.e., computers) and useful to investment professionals. Given the volume, variety, and velocity of available big data, it is important for portfolio managers and investment analysts to have a basic understanding of how unstructured data can be transformed into structured data suitable as inputs to machine learning (ML) methods (in fact, for any type of modeling methods) that can potentially improve their financial forecasts.

This reading describes the steps in using big data, both structured and unstructured, in financial forecasting. The concepts and methods are then demonstrated in a case study of an actual big data project. The project uses text-based data derived from financial documents to train an ML model to classify text into positive or negative sentiment classes for the respective stocks and then to predict sentiment.

Section 2 of the reading covers a description of the key characteristics of big data. Section 3 provides an overview of the steps in executing a financial forecasting project using big data. We then describe in Sections 4–6 key aspects of data preparation and wrangling, data exploration, and model training using structured data and unstructured (textual) data. In Section 7, we bring these pieces together by covering the execution of an actual big data project. A summary in Section 8 concludes the reading.

Big Data in Investment Management

Big data differs from traditional data sources based on the presence of a set of characteristics commonly referred to as the 3Vs: volume, variety, and velocity.

Volume refers to the quantity of data. The US Library of Congress, which is tasked with archiving both digital and physical information artifacts in the United States, has collected hundreds of terabytes of data (one terabyte equals 1,024 gigabytes, which are equal to 1,048,576 megabytes). Several years ago, one of the authors managed an archival project for the Library of Congress in which many terabytes of online content were collected—a copious amount of data at the time. However, in most US industry sectors today, the average company collects more data than the Library of Congress! In big data conversations, terabytes have been replaced with petabytes and exabytes (one exabyte equals 1,024 petabytes, which are equal to 1,048,576 terabytes). The classic grains of sand analogy puts these volumes into perspective: If a megabyte is a tablespoon of sand, then a petabyte is a 1.6-kilometer-long beach and an exabyte is a beach extending about 1,600 kilometers.

Variety pertains to the array of available data sources. Organizations are now dealing with structured, semi-structured, and unstructured data from within and outside the enterprise. Variety includes traditional transactional data; user-generated text, images, and videos; social media; sensor-based data; web and mobile clickstreams; and spatial-temporal data. Effectively leveraging the variety of available data presents both opportunities and challenges, including such legal and ethical issues as data privacy.

Velocity is the speed at which data are created. Many large organizations collect several petabytes of data every hour. With respect to unstructured data, more than one billion new tweets (i.e., a message of 280 characters or less posted on the social media website Twitter) are generated every three days; five billion search queries occur daily. Such information has important implications for real-time predictive analytics in various financial applications. Analyzing such "data-in-motion" poses challenges since relevant patterns and insights might be moving targets relative to situations of "data-at-rest."

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When using big data for inference or prediction, there is a "fourth V": *Veracity relates to the credibility and reliability of different data sources.* Determining the credibility and reliability of data sources is an important part of any empirical investigation. The issue of veracity becomes critically important for big data, however, because of the varied sources of these large datasets. Big data amplifies the age-old challenge of disentangling quality from quantity. Social media, including blogs, forums, and social networking sites, are plagued with spam; by some estimates, as much as 10%–15% of such content is completely fake. Similarly, according to our research, web spam accounts for more than 20% of all content on the worldwide web. Clickstreams from website and mobile traffic are equally susceptible to noise. Furthermore, deriving deep semantic knowledge from text remains challenging in certain instances despite significant advances in natural language processing (NLP).

These Vs have numerous implications for financial technology (commonly referred to as "fintech") pertaining to investment management. Machine learning assessments of creditworthiness, which have traditionally relied on structured financial metrics, are being enhanced by incorporating text derived from financial statements, news articles, and call transcripts. Customers in the financial industry are being segmented based not only on their transactional data but also on their views and preferences expressed on social media (to the degree permissible under applicable privacy agreements). Big data also affords opportunities for enhanced fraud detection and risk management.

EXECUTING A DATA ANALYSIS PROJECT

2

☐ identify and explain steps in a data analysis project

In the era of big data, firms treat data like they do important assets. However, effective big data analytics are critical to allow appropriate data monetization. Let us take financial forecasting as an application area. Numerous forecasting tasks in this domain can benefit from predictive analytics models built using machine learning methods. One common example is predicting whether stock prices (for an individual stock or a portfolio) will go up or down in value at some specific point in the future. Traditionally, financial forecasting relied on various financial and accounting numbers, ratios, and metrics coupled with statistical or mathematical models. More recently, machine learning models have been commonly utilized. However, with the proliferation of textual big data (e.g., online news articles, internet financial forums, social networking platforms), such unstructured data have been shown to offer insights faster (as they are real-time) and have enhanced predictive power.

Textual big data provides several valuable types of information, including topics and sentiment. Topics are what people are talking about (e.g., a firm, an industry, a particular event). Sentiment is how people feel about what they are discussing. For instance, they might express positive, negative, or neutral views (i.e., sentiments) toward a topic of discussion. One study conducted in the United States found that positive sentiment on Twitter could predict the trend for the Dow Jones Industrial Average up to three days later with nearly 87% accuracy.

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Deriving such insights requires supplementing traditional data with textual big data. As depicted in Exhibit 1, the inclusion of big data has immediate implications for building the machine learning model as well as downstream implications for financial forecasting and analysis. We begin with the top half of Exhibit 1, which shows the traditional (i.e., with structured data) *ML Model Building Steps:*

1. *Conceptualization of the modeling task*. This crucial first step entails determining what the output of the model should be (e.g., whether the price of a stock will go up/down one week from now), how this model will be used and by whom, and how it will be embedded in existing or new business processes.
2. *Data collection*. The data traditionally used for financial forecasting tasks are mostly numeric data derived from internal and external sources. Such data are typically already in a structured tabular format, with columns of features, rows of instances, and each cell representing a particular value.
3. *Data preparation and wrangling*. This step involves cleansing and preprocessing of the raw data. Cleansing may entail resolving missing values, out-of-range values, and the like. Preprocessing may involve extracting, aggregating, filtering, and selecting relevant data columns.
4. *Data exploration.* This step encompasses exploratory data analysis, feature selection, and feature engineering.
5. *Model training*. This step involves selecting the appropriate ML method (or methods), evaluating performance of the trained model, and tuning the model accordingly.

Note that these steps are iterative because model building is an iterative process. The insights gained from one iteration may inform the next iteration, beginning with reconceptualization. In contrast with structured data sources, textual big data originating in online news articles, social media, internal/external documents (such as public financial statements), and other openly available data sources are unstructured.

The *TextML Model Building Steps* used for the unstructured data sources of big data are shown in the bottom half of Exhibit 1. They differ from those used for traditional data sources and are typically intended to create output information that is structured. The differences in steps between the text model and traditional model account for the characteristics of big data: volume, velocity, variety, and veracity. In this reading, we mostly focus on the variety and veracity dimensions of big data as they manifest themselves in text. The major differences in the *Text ML Model Building Steps* are in the first four steps:

1. *Text problem formulation*. Analysts begin by determining how to formulate the text classification problem, identifying the exact inputs and outputs for the model. Perhaps we are interested in computing sentiment scores (structured output) from text (unstructured input). Analysts must also decide how the text ML model's classification output will be utilized.
2. *Data (text) curation*. This step involves gathering relevant external text data via web services or **web spidering (scraping or crawling) programs** that extract raw content from a source, typically web pages. Annotation of the text data with high-quality, reliable target (dependent) variable labels might also be necessary for supervised learning and performance evaluation purposes. For instance, experts might need to label whether a given expert assessment of a stock is bearish or bullish.

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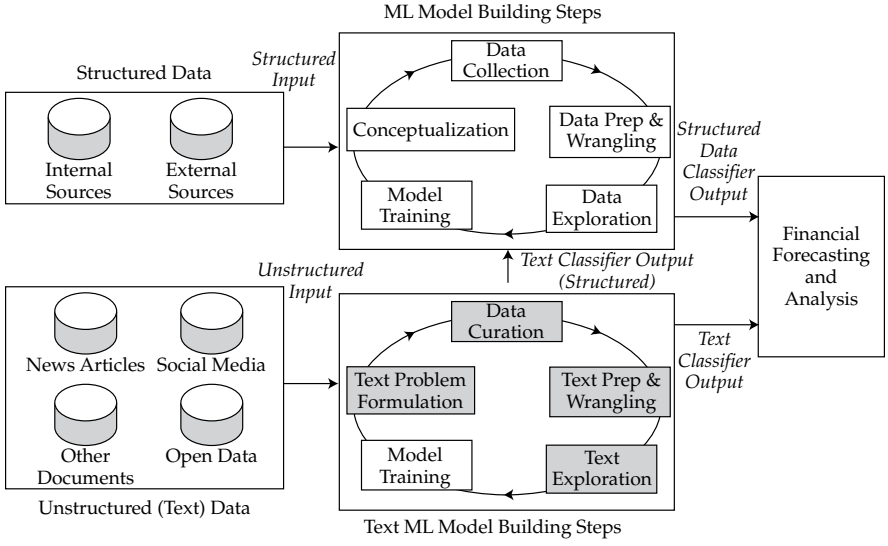
3. *Text preparation and wrangling.* This step involves critical cleansing and preprocessing tasks necessary to convert streams of unstructured data into a format that is usable by traditional modeling methods designed for structured inputs.
4. *Text exploration.* This step encompasses text visualization through techniques, such as word clouds, and text feature selection and engineering.

The resulting output (e.g., sentiment prediction scores) can either be combined with other structured variables or used directly for forecasting and/or analysis.

Next, we describe two key steps from the *ML Model Building Steps* depicted in Exhibit 1 that typically differ for structured data versus textual big data: data/text preparation and wrangling and data/text exploration. We then discuss model training. Finally, we focus on applying these steps to a case study related to classifying and predicting stock sentiment from financial texts.

Exhibit 1: Model Building for Financial Forecasting Using Big Data: Structured (Traditional) vs. Unstructured (Text)



EXAMPLE 1

Steps in ML Model Building

LendALot Corporation is a B2C (business-to-consumer) lender that has traditionally outsourced potential customers' creditworthiness scoring to a third-party firm. Given the recent advances in machine learning (ML)-based "fintech" that goes beyond traditional "repayment history" and "ability to repay" assessments derived from structured data, LendALot would like to develop in-house, ML-based credit scoring capabilities to enhance borrower risk assessment and differentiate itself in the B2C lending market. LendALot would like to follow a phased approach beginning with traditional (structured) data sources and then eventually incorporating textual (unstructured) big data sources. Paul Wang has been asked to lead a new analytics team at LendALot tasked with developing the ML-based creditworthiness scoring model. In the context of machine learning using structured data sources, address the following questions.

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In a later phase of the project, LendALot attempts to improve its credit scoring processes by incorporating textual data in credit scoring. Wang tells his team, "Enhance the creditworthiness scoring model by incorporating insights from text provided by the prospective borrowers in the loan application free response fields."

1. State and explain one decision Wang will need to make related to:

   A. conceptualizing the modeling task.

   B. data collection.

   C. data preparation and wrangling.

   D. data exploration.

   E. model training.

Solution:

A. In the conceptualization step, Wang will need to decide how the output of the ML model will be specified (e.g., a binary classification of creditworthiness), how the model will be used and by whom, and how it will be embedded in LendALot's business processes.

B. In the data collection phase, Wang must decide on what data—internal, external, or both—to use for credit scoring.

C. In the data preparation and wrangling step, Wang will need to decide on data cleansing and preprocessing needs. Cleansing may entail resolving missing values, extreme values, etc. Preprocessing may involve extracting, aggregating, filtering, and selecting relevant data columns.

D. In the data exploration phase, Wang will need to decide which exploratory data analysis methods are appropriate, which features to use in building a credit scoring model, and which features may need to be engineered.

E. In the model training step, Wang must decide which ML algorithm(s) to use. Assuming labeled training data are available, the choice will be among supervised learning algorithms. Decisions will need to be made on how model fit is measured and how the model is validated and tuned.

2. Identify the process step that Wang's statement addresses.

Solution:

Wang's statement relates to the initial step of text problem formulation.

3. State two potential needs of the LendAlot team in relation to text curation.

Solution:

Related to text curation, the team will be using internal data (from loan applications). They will need to ensure that the text comment fields on the loan applications have been correctly implemented and enabled. If these fields are not required, they need to ensure there is a sufficient response rate to analyze.

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4. State two potential needs of the LendAlot team in relation to text preparation and wrangling.

Solution:

Related to text preparation and wrangling, the team will need to carry out the critical tasks of text cleansing and text preprocessing. These two tasks are necessary to convert an unstructured stream of data into structured values for use by traditional modeling methods.

DATA PREPARATION AND WRANGLING

3

- ☐ describe objectives, steps, and examples of preparing and wrangling data
- ☐ evaluate the fit of a machine learning algorithm

Data preparation and wrangling involve cleansing and organizing raw data into a consolidated format. The resulting dataset is suitable to use for further analyses and training a machine learning (ML) model. This is a critical stage, the foundation, in big data projects. Most of the project time is spent on this step, and the quality of the data affects the training of the selected ML model. Domain knowledge—that is, the involvement of specialists in the particular field in which the data are obtained and used—is beneficial and often necessary to successfully execute this step. Data preparation is preceded by data collection, so we discuss the data collection process first.

Before the data collection process even begins, it is important to state the problem, define objectives, identify useful data points, and conceptualize the model. Conceptualization is like a blueprint on a drawing board, a modifiable plan that is necessary to initiate the model building process. A project overview is established by determining the ML model type—supervised or unsupervised—and data sources/collection plans with respect to the needs of the project.

Data collection involves searching for and downloading the raw data from one or multiple sources. Data can be stored in different formats, sources, and locations. As databases are the most common primary sources, building necessary queries with the help of database administrators is critical. Database schemas are built with certain assumptions and exceptions, and it is safest to clarify the database architecture with an administrator or database architect before downloading the necessary data. Data also exist in the form of spreadsheets, comma-separated values (csv) files, text files, and other formats. Care must be taken before using such data, and documentation (often referred to as "Readme" files) must be referred to, if available. **Readme files** are text files provided with the raw data that contain information related to a data file. They are useful for understanding the data and how they can be interpreted correctly.

Alternatively, third-party data vendors can be sources of clean data. External data usually can be accessed through an **application programming interface (API)**—a set of well-defined methods of communication between various software components—or the vendors can deliver the required data in the form of csv files or other formats (as previously mentioned). Using external data can save time and resources that would otherwise go into data preparation and wrangling; however, vendor contracts come with a price. Depending on the big data project constraints, a decision must be made regarding the use of internal or external data based on the trade-offs between time, financial costs, and accuracy. For projects using internal user data, external data

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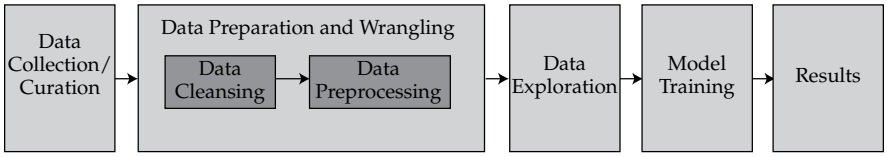
might not be suitable. For example, to understand user traffic on a company website, internally recorded site visits and click frequency may be captured and stored in the internal databases. External data are advantageous when a project requires generic data, such as demographics of a geographic area or traffic data of a public service. Another consideration in using external vendor provided data is that during the cleansing process, underlying trends in the data that are important for particular end-uses may be masked or even lost. This is where "alpha" is often found; so by simply buying a dataset from a vendor, you may lose your information edge. Of course, application of the data (e.g., merging and combining, putting through different types of models) will be different for everyone who uses it; there are always different ways to extract value.

Once the data are collected, the data preparation and wrangling stage begins. This stage involves two important tasks: cleansing and preprocessing, respectively. Exhibit 2 outlines data preparation and wrangling and defines the two component tasks. These tasks are explained in detail under the structured and unstructured sub-sections because the steps vary by the nature of data.

Exhibit 2: Data Preparation and Wrangling Stage



Data Preparation (Cleansing): This is the initial and most common task in data preparation that is performed on raw data. Data cleansing is the process of examining, identifying, and mitigating errors in raw data. Normally, the raw data are neither sufficiently complete nor sufficiently clean to directly train the ML model. Manually entered data can have incomplete, duplicated, erroneous, or inaccurate values. Automated data (recorded by systems) can have similar problems due to server failures and software bugs.

Data Wrangling (Preprocessing): This task performs transformations and critical processing steps on the cleansed data to make the data ready for ML model training. Raw data most commonly are not present in the appropriate format for model consumption. After the cleansing step, data need to be processed by dealing with outliers, extracting useful variables from existing data points, and scaling the data.

Structured Data

Data Preparation (Cleansing)

Structured data are organized in a systematic format that is readily searchable and readable by computer operations for processing and analyzing. In structured data, data errors can be in the form of incomplete, invalid, inaccurate, inconsistent, non-uniform, and duplicate data observations. The data cleansing process mainly deals with identifying and mitigating all such errors. Exhibit 3 shows a raw dataset before cleansing. The data have been collected from different sources and are organized in a data matrix (or data table) format. Each row contains observations of each customer of a US-based bank. Each column represents a variable (or feature) corresponding to each customer.

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Exhibit 3: Raw Data Before Cleansing

1	ID	Name	Gender	Date of Birth	Salary	Other Income	State	Credit Card
2	1	Mr. ABC	M	12/5/1970	$50,200	$5,000	VA	Y
3	2	Ms. XYZ	M	15 Jan, 1975	$60,500	$0	NY	Y
4	3	EFG		1/13/1979	$65,000	$1,000	CA	N
5	4	Ms. MNO	F	1/1/1900	—	—	FL	Don't Know
6	5	Ms. XYZ	F	15/1/1975	$60,500	$0		Y
7	6	Mr. GHI	M	9/10/1942	NA	$55,000	TX	N
8	7	Mr. TUV	M	2/27/1956	$300,000	$50,000	CT	Y
9	8	Ms. DEF	F	4/4/1980	$55,000	$0	British Columbia	N

The possible errors in a raw dataset include the following:

1. *Incompleteness error* is where the data are not present, resulting in missing data. This can be corrected by investigating alternate data sources. Missing values and NAs (not applicable or not available values) must be either omitted or replaced with "NA" for deletion or substitution with imputed values during the data exploration stage. The most common imputations are mean, median, or mode of the variable or simply assuming zero. In Exhibit 3, rows 4 (ID 3), 5 (ID 4), 6 (ID 5), and 7 (ID 6) are incomplete due to missing values in either Gender, Salary, Other Income, Name (Salutation), and State columns.
2. *Invalidity error* is where the data are outside of a meaningful range, resulting in invalid data. This can be corrected by verifying other administrative data records. In Exhibit 3, row 5 likely contains invalid data as the date of birth is out of the range of the expected human life span.
3. *Inaccuracy error* is where the data are not a measure of true value. This can be rectified with the help of business records and administrators. In Exhibit 3, row 5 is inaccurate (it shows "Don't Know"); in reality, every person either has a credit card or does not.
4. *Inconsistency error* is where the data conflict with the corresponding data points or reality. This contradiction should be eliminated by clarifying with another source. In Exhibit 3, row 3 (ID 2) is likely to be inconsistent as the Name column contains a female title and the Gender column contains male.
5. *Non-uniformity error* is where the data are not present in an identical format. This can be resolved by converting the data points into a preferable standard format. In Exhibit 3, the data under the Date of Birth column is present in various formats. The data under the Salary column may also be non-uniform as the monetary units are ambiguous; the dollar symbol can represent US dollar, Canadian dollar, or others.
6. *Duplication error* is where duplicate observations are present. This can be corrected by removing the duplicate entries. In Exhibit 3, row 6 is a duplicate as the data under Name and Date of Birth columns are identical to the ones in row 3, referring to the same customer.

Exhibit 4 shows the dataset after completion of the cleansing process.

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Exhibit 4: Data After Cleansing

1	ID	Name	Gender	Date of Birth	Salary	Other Income	State	Credit Card
2	1	Mr. ABC	M	12/5/1970	USD 50200	USD 5000	VA	Y
3	2	Ms. XYZ	F	1/15/1975	USD 60500	USD 0	NY	Y
4	3	Mr. EFG	M	1/13/1979	USD 65000	USD 1000	CA	N
5	6	Mr. GHI	M	9/10/1942	USD 0	USD 55000	TX	N
6	7	Mr. TUV	M	2/27/1956	USD 300000	USD 50000	CT	Y
7	8	Ms. DEF	F	4/4/1980	CAD 55000	CAD 0	British Columbia	N

Data cleansing can be expensive and cumbersome because it involves the use of automated, rule-based, and pattern recognition tools coupled with manual human inspection to sequentially check for the aforementioned types of errors row by row and column by column. The process involves a detailed data analysis as an initial step in identifying various errors that are present in the data. In addition to a manual inspection and verification of the data, analysis software, such as SPSS, can be used to understand **metadata** (data that describes and gives information about other data) about the data properties to use as a starting point to investigate any errors in the data. The business value of the project determines the necessary quality of data cleansing and subsequently the amount of resources used in the cleansing process. In case the errors cannot be resolved due to lack of available resources, the data points with errors can simply be omitted depending on the size of the dataset. For instance, if a dataset is large with more than 10,000 rows, removing a few rows (approximately 100) may not have a significant impact on the project. If a dataset is small with less than 1,000 rows, every row might be important and deleting many rows thus harmful to the project.

Data Wrangling (Preprocessing)

To make structured data ready for analyses, the data should be preprocessed. Data preprocessing primarily includes transformations and scaling of the data. These processes are exercised on the cleansed dataset. The following transformations are common in practice:

1. *Extraction:* A new variable can be extracted from the current variable for ease of analyzing and using for training the ML model. In Exhibit 4, the Date of Birth column consists of dates that are not directly suitable for analyses. Thus, an additional variable called "Age" can be extracted by calculating the number of years between the present day and date of birth.
2. *Aggregation:* Two or more variables can be aggregated into one variable to consolidate similar variables. In Exhibit 4, the two forms of income, Salary and Other Income, can be summed into a single variable called Total Income.
3. *Filtration:* The data rows that are not needed for the project must be identified and filtered. In Exhibit 4, row 7 (ID 8) has a non-US state; however, this dataset is for the US-based bank customers where it is required to have a US address.
4. *Selection:* The data columns that are intuitively not needed for the project can be removed. This should not be confused with feature selection, which is explained later. In Exhibit 4, Name and Date of Birth columns are not

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required for training the ML model. The ID column is sufficient to identify the observations, and the new extracted variable Age replaces the Date of Birth column.

5. *Conversion:* The variables can be of different types: nominal, ordinal, continuous, and categorical. The variables in the dataset must be converted into appropriate types to further process and analyze them correctly. This is critical for ML model training. Before converting, values must be stripped out with prefixes and suffixes, such as currency symbols. In Exhibit 4, Name is nominal, Salary and Income are continuous, Gender and Credit Card are categorical with 2 classes, and State is nominal. In case row 7 is not excluded, the Salary in row 7 must be converted into US dollars. Also, the conversion task applies to adjusting time value of money, time zones, and others when present.

Outliers may be present in the data, and domain knowledge is needed to deal with them. Any outliers that are present must first be identified. The outliers then should be examined and a decision made to either remove or replace them with values imputed using statistical techniques. In Exhibit 4, row 6 (ID 7) is an outlier because the Salary value is far above the upper quartile. Row 5 (ID 6) is also an outlier because the Salary value is far below the lower quartile. However, after the aggregation and formation of a new variable Total Income, as shown in Exhibit 5, row 5 (ID 6), it is no longer an outlier.

In practice, several techniques can be used to detect outliers in the data. Standard deviation can be used to identify outliers in normally distributed data. In general, a data value that is outside of 3 standard deviations from the mean may be considered an outlier. The interquartile range (IQR) can be used to identify outliers in data with any form of distribution. IQR is the difference between the 75th and the 25th percentile values of the data. In general, data values outside of the following are considered as outliers: +1.5 x IQR + 3rd Quartile Upper Bound; and -1.5 x IQR + 2nd Quartile Lower Bound. Using a multiple of 3.0 (instead of 1.5) times IQR would indicate extreme values.

There are several practical methods for handling outliers. When extreme values and outliers are simply removed from the dataset, it is known as **trimming** (also called truncation). For example, a 5% trimmed dataset is one for which the 5% highest and the 5% lowest values have been removed. When extreme values and outliers are replaced with the maximum (for large value outliers) and minimum (for small value outliers) values of data points that are not outliers, the process is known as **winsorization**.

Exhibit 5: Data After Applying Transformations

1	ID	Gender	Age	Total Income	State	Credit Card
2	1	M	48	55200	VA	Y
3	2	F	43	60500	NY	Y
4	3	M	39	66000	CA	N
5	6	M	76	55000	TX	N

Scaling is a process of adjusting the range of a feature by shifting and changing the scale of data. Variables, such as age and income, can have a diversity of ranges that result in a heterogeneous training dataset. For better ML model training when using such methods as support vector machines (SVMs) and artificial neural networks (ANNs),

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all variables should have values in the same range to make the dataset homogeneous. It is important to remove outliers before scaling is performed. Here are two of the most common ways of scaling:

1. *Normalization* is the process of rescaling numeric variables in the range of [0, 1]. To normalize variable X, the minimum value (X_{min}) is subtracted from each observation (X_i), and then this value is divided by the difference between the maximum and minimum values of X ($X_{\text{max}} - X_{\text{min}}$) as follows:

$$X_{i\ (\text{normalized})} = \frac{X_i - X_{\text{min}}}{X_{\text{max}} - X_{\text{min}}}. \quad (1)$$

2. *Standardization* is the process of both centering and scaling the variables. Centering involves subtracting the mean (μ) of the variable from each observation (X_i) so the new mean is 0. Scaling adjusts the range of the data by dividing the centered values ($X_i - \mu$) by the standard deviation (σ) of feature X. The resultant standardized variable will have an arithmetic mean of 0 and standard deviation of 1.

$$X_{i\ (\text{standardized})} = \frac{X_i - \mu}{\sigma} \quad (2)$$

Normalization is sensitive to outliers, so treatment of outliers is necessary before normalization is performed. Normalization can be used when the distribution of the data is not known. Standardization is relatively less sensitive to outliers as it depends on the mean and standard deviation of the data. However, the data must be normally distributed to use standardization.

EXAMPLE 2

Preparing and Wrangling Structured Data

Paul Wang's analytics team at LendALot Corporation is working to develop its first ML model for classifying prospective borrowers' creditworthiness. Wang has asked one of his data scientists, Lynn Lee, to perform a preliminary assessment of the data cleansing and preprocessing tasks the team will need to perform. As part of this assessment, Lee pulled the following sample of data for manual examination, which she brings to Wang to discuss.

1	ID	Name	Loan Outcome	Income (USD)	Loan Amount (USD)	Credit Score	Loan Type
2	1	Mr. Alpha	No Default	34,000	10,000	685	Mortgage
3	2	Ms. Beta	No Default	–63,050	49,000	770	Student Loan
4	3	Mr. Gamma	Defaulted	20,565	35,000	730	
5	4	Ms. Delta	No Default	50,021	unknown	664	Mortgage
6	5	Mr. Epsilon	Defaulted	100,350	129,000	705	Car Loan
7	6	Mr. Zeta	No Default	800,000	300,000	800	Boat Loan
8	6	Mr. Zeta	No Default	800,000	300,000	800	Boat Loan

After sharing a concern that the data should be thoroughly cleansed, Wang makes the following statements:

Statement 1 "Let's keep the ID column and remove the column for Name from the dataset."

Statement 2 "Let's create a new feature, "Loan Amount as a Percent of Income," to use as an additional feature."

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1. The data shown for Ms. Beta contain what is *best described* as an:

   A. invalidity error.
   B. inaccuracy error.
   C. incompleteness error.

Solution:

A is correct. This is an invalidity error because the data are outside of a meaningful range. Income cannot be negative.

2. The data shown for Mr. Gamma contain what is *best described* as an:

   A. invalidity error.
   B. duplication error.
   C. incompleteness error.

Solution:

C is correct. This is an incompleteness error as the loan type is missing.

3. The data shown for Ms. Delta contain what is *best described* as an:

   A. invalidity error.
   B. inaccuracy error.
   C. duplication error.

Solution:

B is correct. This is an inaccuracy error because LendALot must know how much they have lent to that particular borrower (who eventually repaid the loan as indicated by the loan outcome of no default).

4. The data shown for Mr. Zeta contain what is *best described* as an:

   A. invalidity error.
   B. inaccuracy error.
   C. duplication error.

Solution:

C is correct. Row 8 duplicates row 7: This is a duplication error.

5. The process mentioned in Wang's first statement is *best described* as:

   A. data selection.
   B. data extraction.
   C. data engineering

Solution:

A is correct. The process mentioned involves selecting the data to use. The proposal makes sense; with "ID," "Name" is not needed to identify an observation.

6. Wang's second statement is *best described* as:

   A. data selection.

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B. data extraction.

C. data engineering.

Solution:

B is correct. The proposed feature is a ratio of two existing data. *Data extraction* is the process of creating (i.e., extracting) new variables from existing ones in the data.

4 UNSTRUCTURED (TEXT) DATA

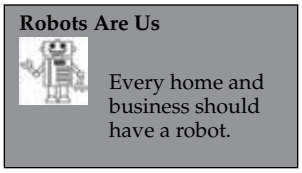
☐ describe objectives, steps, and examples of preparing and wrangling data

Unstructured data are not organized into any systematic format that can be processed by computers directly. They are available in formats meant for human usage rather than computer processing. Unstructured data constitute approximately 80% of the total data available today. They can be in the form of text, images, videos, and audio files. Unlike in structured data, preparing and wrangling unstructured data are both more challenging. For analysis and use to train the ML model, the unstructured data must be transformed into structured data. In this section, text data will be used to demonstrate unstructured data preparation and wrangling. The cleansing and preprocessing of text data is called *text processing*. Text processing is essentially cleansing and transforming the unstructured text data into a structured format. Text processing can be divided into two tasks: cleansing and preprocessing. The following content is related to text data in the English language.

Text Preparation (Cleansing)

Raw text data are a sequence of characters and contain other non-useful elements, including html tags, punctuations, and white spaces (including tabs, line breaks, and new lines). It is important to clean the text data before preprocessing. Exhibit 6 shows a sample text from the home page for the hypothetical company Robots Are Us website. The text appears to be clean visually and is designed for human readability.

Exhibit 6: Sample Text from Robots Are Us Home Page



However, the source text that can be downloaded is not as clean. The raw text contains html tags and formatting elements along with the actual text. Exhibit 7 shows the raw text from the source.

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Exhibit 7: Raw Text from the Source

```
<h1 class="text-left mb-3">Robots Are Us</h1>
<h2> Every home and business should have a robot    </h2>
```

The initial step in text processing is cleansing, which involves basic operations to clean the text by removing unnecessary elements from the raw text. Text operations often use regular expressions. A **regular expression (regex)** is a series that contains characters in a particular order. Regex is used to search for patterns of interest in a given text. For example, a regex "<.*?>" can be used to find all the html tags that are present in the form of <…> in text.[1] GREP (global regular expression print) is a commonly available utility in programming languages for searching patterns using regex. Once a pattern is found, it can be removed or replaced. Additionally, advanced html parsers and packages are available in the popular programming languages, such as R and Python, to deal with this task.

The following steps describe the basic operations in the text cleansing process.

1. *Remove html tags*: Most of the text data are acquired from web pages, and the text inherits html markup tags with the actual content. The initial task is to remove (or strip) the html tags that are not part of the actual text using programming functions or using regular expressions. In Exhibit 7, </h2> is an html tag that can be identified by a regex and be removed. Note that it is not uncommon to keep some generic html tags to maintain certain formatting meaning in the text.
2. *Remove Punctuations*: Most punctuations are not necessary for text analysis and should be removed. However, some punctuations, such as percentage signs, currency symbols, and question marks, may be useful for ML model training. These punctuations should be substituted with such annotations as /percentSign/, /dollarSign/, and /questionMark/ to preserve their grammatical meaning in the text. Such annotations preserve the semantic meaning of important characters in the text for further text processing and analysis stages. It is important to note that periods (dots) in the text need to be processed carefully. There are different circumstances for periods to be present in text—characteristically used for abbreviations, sentence boundaries, and decimal points. The periods and the context in which they are used need to be identified and must be appropriately replaced or removed. In general, periods after abbreviations can be removed, but the periods separating sentences should be replaced by the annotation /endSentence/. Some punctuations, such as hyphens and underscores, can be kept in the text to keep the consecutive words intact as a single term (e.g., e-mail). Regex are often used to remove or replace punctuations.
3. *Remove Numbers*: When numbers (or digits) are present in the text, they should be removed or substituted with an annotation /number/. This helps inform the computer that a number is present, but the actual value of the number itself is not helpful for categorizing/analyzing the text. Such operations are critical for ML model training. Otherwise, the computers will treat each number as a separate word, which may complicate the analyses or add noise. Regex are often used to remove or replace numbers. However, the number and any decimals must be retained where the outputs of interest

1 A regex of the form "<.*?>" will identify all html tags with anything (*) of any length (?) between the brackets (< >).

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are the actual values of the number. One such text application is information extraction (IE), where the goal is to extract relevant information from a given text. An IE task could be extracting monetary values from financial reports, where the actual number values are critical.

4. *Remove white spaces*: It is possible to have extra white spaces, tab spaces, and leading and ending spaces in the text. The extra white spaces may be introduced after executing the previously mentioned operations. These should be identified and removed to keep the text intact and clean. Certain functions in programming languages can be used to remove unnecessary white spaces from the text. For example, the text mining package in R offers a *stripwhitespace* function.

Exhibit 8 uses a sample financial text to show the transformations occurring after applying each operation of the text cleansing process. The four steps are applied on a mock financial text after scraping from a source. As noted previously, scraping (or web scraping) is a technique to extract raw content from a source, typically web pages. It is important to note that the sequence and choice of cleansing operations does matter. For instance, after removing punctuation, the "1.2 million" becomes "12 million." This is acceptable here since a subsequent operation replaces all numbers with a "/number/" tag. However, if numbers were not replaced with such tags, the punctuation removal operation could affect the data.

Exhibit 8: Text Cleansing Process Example

Original text from a financial statement as shown on a webpage

CapEx on the normal operations remained stable on historicallylow levels, $800,000 compared to $1.2 million last year.

Quarter 3, so far, is 5% sales growth quarter-to-date, and year-to-date, we have a 4% local currency sales development.

Raw text after scraping from the source

 CapEx on the normal operations remained stable on historically low levels, $800,000 compared to $1.2 million last year. <b/><b/> Quarter 3, so far, is 5% sales growth quarter-to-date, and year-to-date, we have a 4% local currency sales development.

Text after removing html tags

CapEx on the normal operations remained stable on historically low levels, $800,000 compared to $1.2 million last year.
Quarter 3, so far, is 5% sales growth quarter-to-date, and year-to-date, we have a 4% local currency sales development.

Text after removing and replacing punctuations

CapEx on the normal operations remained stable on historically low levels /dollarSign/800000 compared to /dollarSign/12 million last year /endSentence/ Quarter 3 so far is 5 /percentSign/ sales growth quarter-to-date and year-to-date we have a 4 /percentSign/ local currency sales development /endSentence/

Text after replacing numbers

CapEx on the normal operations remained stable on historically low levels /dollarSign//number / compared to/dollarSign//number/ million last year /endSentence/ Quarter/number/ so far is /number/ /percentSign/sales growth quarter-to-date and year-to-date we have a /number/ / percentSign/ local currency sales development /endSentence/

Text after removing extra white spaces

CapEx on the normal operations remained stable on historically low levels/dollarSign//number /compared to/dollarSign//number/million last year/endSentence/ Quarter/number/so far is /number//percentSign/sales growth quarter-to-date and year-to-date we have a/number// percentSign/local currency sales development/endSentence/

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Text Wrangling (Preprocessing)

To further understand text processing, tokens and tokenization need to be defined. A **token** is equivalent to a word, and **tokenization** is the process of splitting a given text into separate tokens. In other words, a text is considered to be a collection of tokens. Tokenization can be performed at word or character level, but it is most commonly performed at word level. Exhibit 9 shows a sample dataset of four cleansed texts and their word tokens.

Exhibit 9: Tokenization of Four Texts

	Cleaned Texts	Tokens
Text 1	The man went to the market today	The man went to the market today
Text 2	Market values are increasing	Market values are increasing
Text 3	Increased marketing is needed	Increased marketing is needed
Text 4	There is no market for the product	There is no market for the product

Similar to structured data, text data also require normalization. The normalization process in text processing involves the following:

1. *Lowercasing* the alphabet removes distinctions among the same words due to upper and lower cases. This action helps the computers to process the same words appropriately (e.g., "The" and "the").
2. *Stop words* are such commonly used words as "the," "is," and "a." Stop words do not carry a semantic meaning for the purpose of text analyses and ML training. However, depending on the end-use of text processing, for advance text applications it may be critical to keep the stop words in the text in order to understand the context of adjacent words. For ML training purposes, stop words typically are removed to reduce the number of tokens involved in the training set. A predefined list of stop words is available in programming languages to help with this task. In some cases, additional stop words can be added to the list based on the content. For example, the word "exhibit" may occur often in financial filings, which in general is not a stop word but in the context of the filings can be treated as a stop word.
3. *Stemming* is the process of converting inflected forms of a word into its base word (known as stem). Stemming is a rule-based approach, and the results need not necessarily be linguistically sensible. Stems may not be the same as the morphological root of the word. Porter's algorithm is the most popular method for stemming. For example, the stem of the words "analyzed" and "analyzing" is "analyz." Similarly, the British English variant "analysing" would become "analys." Stemming is available in R and Python. The text mining package in R provides a *stemDocument* function that uses this algorithm.
4. *Lemmatization* is the process of converting inflected forms of a word into its morphological root (known as lemma). Lemmatization is an algorithmic approach and depends on the knowledge of the word and language structure. For example, the lemma of the words "analyzed" and "analyzing" is "analyze." Lemmatization is computationally more expensive and advanced.

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Stemming or lemmatization will reduce the repetition of words occurring in various forms and maintain the semantic structure of the text data. Stemming is more common than lemmatization in the English language since it is simpler to perform. In text data, data sparseness refers to words that appear very infrequently, resulting in data consisting of many unique, low frequency tokens. Both techniques decrease data sparseness by aggregating many sparsely occurring words in relatively less sparse stems or lemmas, thereby aiding in training less complex ML models.

After the cleansed text is normalized, a bag-of-words is created. **Bag-of-words (BOW)** representation is a basic procedure used to analyze text. It is essentially a collection of a distinct set of tokens from all the texts in a sample dataset. BOW is simply a set of words and does not capture the position or sequence of words present in the text. However, it is memory efficient and easy to handle for text analyses.

Exhibit 10 shows the BOW and transformations occurring in each step of normalization on the cleansed texts from Exhibit 9. Note that the number of words decreases as the normalizing steps are applied, making the resulting BOW smaller and simpler.

Exhibit 10: Bag-of-Words Representation of Four Texts Before and After Normalization Process

BOW before normalizing

"The"	"man"	"went"	"to"	"the"	"market"
"today"	"Market"	"values"	"are"	"increasing"	"Increased"
"marketing"	"is"	"needed"	"There"	"no"	"for"
"product"					

BOW after removing uppercase letters

"the"	"man"	"went"	"to"	"market"	"today"
"values"	"are"	"increasing"	"increased"	"marketing"	"is"
"needed"	"there"	"no"	"for"	"product"	

BOW after removing stop words

"man"	"went"	"market"	"today"	"values"	"increasing"
"increased"	"marketing"	"needed"	"product"		

BOW after stemming

"man"	"went"	"market"	"today"	"valu"	"increas"	"need"	"product"

The last step of text preprocessing is using the final BOW after normalizing to build a **document term matrix (DTM)**. DTM is a matrix that is similar to a data table for structured data and is widely used for text data. Each row of the matrix belongs to a document (or text file), and each column represents a token (or term). The number of rows of DTM is equal to the number of documents (or text files) in a sample dataset. The number of columns is equal to the number of tokens from the BOW that is built using all the documents in a sample dataset. The cells can contain the counts of the number of times a token is present in each document. The matrix cells can be filled with other values that will be explained in the financial forecasting project section of this reading; a large dataset is helpful in understanding the concepts. At this point, the unstructured text data are converted to structured data that can be processed further and used to train the ML model. Exhibit 11 shows a DTM constructed from the resultant BOW of the four texts from Exhibit 10.

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Exhibit 11: DTM of Four Texts and Using Normalized BOW Filled with Counts of Occurrence

	man	went	market	today	valu	increas	need	product
Text 1	1	1	1	1	0	0	0	0
Text 2	0	0	1	0	1	1	0	0
Text 3	0	0	1	0	0	1	1	0
Text 4	0	0	1	0	0	0	0	1

As seen in Exhibit 10, BOW does not represent the word sequences or positions, which limits its use for some advanced ML training applications. In the example, the word "no" is treated as a single token and has been removed during the normalization because it is a stop word. Consequently, this fails to signify the negative meaning ("no market") of the text (i.e., Text 4). To overcome such problems, a technique called n-grams can be employed. **N-grams** is a representation of word sequences. The length of a sequence can vary from 1 to *n*. When one word is used, it is a unigram; a two-word sequence is a bigram; and a 3-word sequence is a trigram; and so on. Exhibit 10, for example, shows a unigram (n = 1) BOW. The advantage of n-grams is that they can be used in the same way as unigrams to build a BOW. In practice, different n-grams can be combined to form a BOW and eventually be used to build a DTM. Exhibit 12 shows unigrams, bigrams, and trigrams. Exhibit 12 also shows a combined unigram-to-trigram BOW for the particular text. Stemming can be applied on the cleansed text before building n-grams and BOW (not shown in Exhibit 12).

Exhibit 12: N-Grams and N-Grams BOW

Clean text

The man went to the market today

Unigrams

"The" "man" "went" "to" "the" "market" "today"

Bigrams

"The_man" "man_went" "went_to" "to_the" "the_market" "market_today"

Trigrams

"The_man_went" "man_went_to" "went_to_the" "to_the_market" "the_market_today"

BOW before normalizing

"The" "man" "went" "to" "the" "market" "today"
"The_man" "man_went" "went_to" "to_the" "the_market" "market_today" "The_man_went"
"man_went_to" "went_to_the" "to_the_market" "the_market_today"

BOW after removing upper case letters

"the" "man" "went" "to" "market" "today" "the_man"
"man_went" "went_to" "to_the" "the_market" "market_today" "the_man_went" "man_went_to"
"went_to_the" "to_the_market" "the_market_today"

BOW after removing stop words

"man" "went" "market" "today" "the_man" "man_went" "went_to"
"to_the" "the_market" "market_today" "the_man_went" "man_went_to" "went_to_the" "to_the_market"
"the_market_today"

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The n-grams implementation will vary the impact of normalization on the BOW. Even after removing isolated stop words, stop words tend to persist when they are attached to their adjacent words. For instance, "to_the" (Exhibit 12) is a single bigram token consisting of stop words and will not be removed by the predetermined list of stop words.

EXAMPLE 3

Unstructured Data Preparation and Wrangling

1. The output produced by preparing and wrangling textual data is best described as a:

   A. data table.

   B. confusion matrix.

   C. document term matrix.

Solution:

C is correct. The objective of data preparation and wrangling of textual data is to transform the unstructured data into structured data. The output of these processes is a document term matrix that can be read by computers. The document term matrix is similar to a data table for structured data.

2. In text cleansing, situations in which one may need to add an annotation include the removal of:

   A. html tags.

   B. white spaces.

   C. punctuations.

Solution:

C is correct. Some punctuations, such as percentage signs, currency symbols, and question marks, may be useful for ML model training, so when such punctuations are removed annotations should be added.

3. A column of a document term matrix is *best* described as representing:

   A. a token.

   B. a regularization term.

   C. an instance.

Solution:

A is correct. Each column of a document term matrix represents a token from the bag-of-words that is built using all the documents in a sample dataset.

4. A cell of a document term matrix is *best* described as containing:

   A. a token.

   B. a count of tokens.

   C. a count of instances.

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Solution:

B is correct. A cell in a document term matrix contains a count of the number of tokens of the kind indicated in the column heading.

5. Points to cover in normalizing textual data include:

   A. removing numbers.

   B. removing white spaces.

   C. lowercasing the alphabet.

Solution:

C is correct. The other choices are related to text cleansing.

6. When some words appear very infrequently in a textual dataset, techniques that may address the risk of training highly complex models include:

   A. stemming.

   B. scaling.

   C. data cleansing.

Solution:

A is correct. Stemming, the process of converting inflected word forms into a base word (or stem), is one technique that can address the problem described.

7. Which of the following statements concerning tokenization is *most* accurate?

   A. Tokenization is part of the text cleansing process.

   B. Tokenization is most commonly performed at the character level.

   C. Tokenization is the process of splitting a given text into separate tokens.

Solution:

C is correct, by definition. The other choices are not true.

DATA EXPLORATION OBJECTIVES AND METHODS

5

☐ describe objectives, methods, and examples of data exploration

Data exploration is a crucial part of big data projects. The prepared data are explored to investigate and comprehend data distributions and relationships. The knowledge that is gained about the data in this stage is used throughout the project. The outcome and quality of exploration strongly affects ML model training results. Domain knowledge plays a vital role in exploratory analysis as this stage should involve cooperation between analysts, model designers, and experts in the particular data domain. Data exploration without domain knowledge can result in ascertaining spurious relationships among the variables in the data that can mislead the analyses. The data exploration stage follows the data preparation stage and leads to the model training stage.

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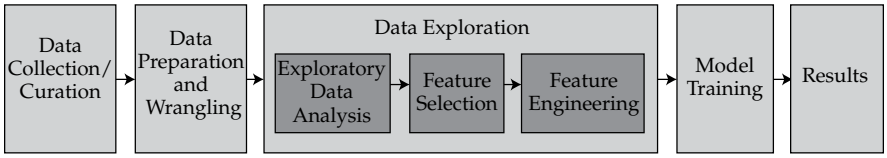
Data exploration involves three important tasks: exploratory data analysis, feature selection, and feature engineering. These three tasks are outlined in Exhibit 13 and are defined and further explained under the structured and unstructured data subsections.

Exhibit 13: Data Exploration Stage



Exploratory data analysis (EDA) is the preliminary step in data exploration. Exploratory graphs, charts, and other visualizations, such as heat maps and word clouds, are designed to summarize and observe data. In practice, many exploratory graphs are made for investigation and can be made swiftly using statistical programming and generic spreadsheet software tools. Data can also be summarized and examined using quantitative methods, such as descriptive statistics and central tendency measures. An important objective of EDA is to serve as a communication medium among project stakeholders, including business users, domain experts, and analysts. Relatively quick and easy exploratory visualizations help stakeholders connect and ensure the prepared data are sensible. Other objectives of EDA include:

- understanding data properties,
- finding patterns and relationships in data,
- inspecting basic questions and hypotheses,
- documenting data distributions and other characteristics, and
- planning modeling strategies for the next steps.

Feature selection is a process whereby only pertinent features from the dataset are selected for ML model training. Selecting fewer features decreases ML model complexity and training time. **Feature engineering** is a process of creating new features by changing or transforming existing features. Model performance heavily depends on feature selection and engineering.

Structured Data

Exploratory Data Analysis

For structured data, each data table row contains an observation and each column contains a feature. EDA can be performed on a single feature (one-dimension) or on multiple features (multi-dimension). For high-dimension data with many features, EDA can be facilitated by using a dimension reduction technique, such as principal components analysis (PCA). Based on the number of dimensions, the exploratory techniques will vary.

For one-dimensional data, summary statistics, such as mean, median, quartiles, ranges, standard deviations, skewness, and kurtosis, of a feature can be computed. One-dimension visualization summarizes each feature in the dataset. The basic one-dimension exploratory visualizations are as follows:

- Histograms
- Bar charts

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- Box plots
- Density plots

Histograms represent equal bins of data and their respective frequencies. They can be used to understand the high-level distribution of the data. Bar charts summarize the frequencies of categorical variables. Box plots show the distribution of continuous data by highlighting the median, quartiles, and outliers of a feature that is normally distributed. Density plots are another effective way to understand the distribution of continuous data. Density plots are smoothed histograms and are commonly laid on top of histograms, as shown in Exhibit 14. This histogram shows a hypothetical annual salary distribution (in £) of entry-level analyst positions at UK banks. The data represent a normal distribution with an approximate mean of £68,500.

Exhibit 14: Histogram with Superimposed Density Plot

For data with two or more dimensions, summary statistics of relationships, such as a correlation matrix, can be calculated. Two- or more-dimensional visualization explores interactions between different features in the dataset. Common methods include scatterplots and line graphs. In multi-dimensional visualization, one-dimensional plots are overlaid to summarize each feature, thus enabling comparison between features. Additionally, attributes (e.g., color, shape, and size) and legends can be used creatively to pack more information about the data into fewer graphs.

For multivariate data, commonly utilized exploratory visualization designs include stacked bar and line charts, multiple box plots, and scatterplots showing multivariate data that use different colors or shapes for each feature. Multiple box plots can be arranged in a single chart, where each individual box plot represents a feature. Such a multi-box plot chart assesses the relationship between each feature (x-axis) in the dataset and the target variable of interest (y-axis). The multi-box plot chart in Exhibit 15 represents units of shares purchased versus stock price for a hypothetical stock. The x-axis shows the stock price in increments of $0.125, and the y-axis shows units of shares purchased. The individual box plots indicate the distribution of shares purchased at the different stock prices. When the stock price is $0.25, the median number of shares purchased is the highest; when the stock price is $0.625, the median number of shares purchased is the lowest. However, visually it appears that the number of shares purchased at different stock prices is not significantly different.

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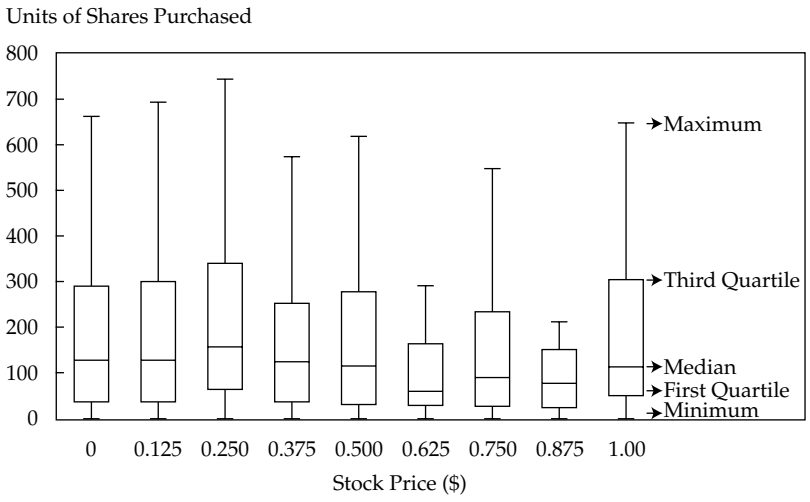
Exhibit 15: Multiple Box Plots in One Chart



Two-dimensional charts can summarize and approximately measure relationships between two or more features. An example scatterplot in Exhibit 16 shows the interaction of two hypothetical features: age (x-axis) and annual salary (y-axis). The feature on the y-axis tends to increase as the feature on the x-axis increases. This pattern appears true visually; however, it may not be a statistically significant relationship. A scatterplot provides a starting point where relationships can be examined visually. These potential relationships should be tested further using statistical tests. Common parametric statistical tests include ANOVA, *t*-test, and Pearson correlation. Common non-parametric statistical tests include chi-square and the Spearman rank-order correlation.

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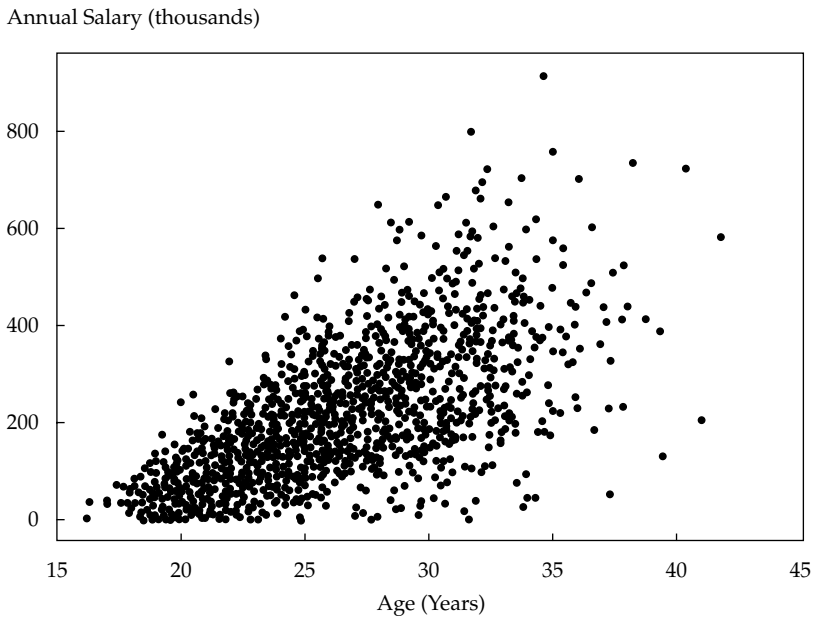
Exhibit 16: Scatterplot Showing a Linear Relationship Between Two Features



In addition to visualization, descriptive statistics are a good means to summarize data. Central tendency measures as well as minimum and maximum values for continuous data are useful. Counts and frequencies for categorical data are commonly employed to gain insight regarding the distribution of possible values.

EDA is not only useful for revealing possible relationships among features or general trends in the data; it is also beneficial during the feature selection and engineering stages. These possible relationships and trends in the data may be used to suggest new features that, when incorporated into a model, may improve model training.

Feature Selection

Structured data consist of features, represented by different columns of data in a table or matrix. After using EDA to discover relevant patterns in the data, it is essential to identify and remove unneeded, irrelevant, and redundant features. Basic diagnostic testing should also be performed on features to identify redundancy, heteroscedasticity, and multi-collinearity. The objective of the feature selection process is to assist in identifying significant features that when used in a model retain the important patterns and complexities of the larger dataset while requiring fewer data overall. This last point is important since computing power is not free (i.e., explicit costs and processing time).

Typically, structured data even after the data preparation step can contain features that do not contribute to the accuracy of an ML model or that negatively affect the quality of ML training. The most desirable outcome is a parsimonious model with fewer features that provides the maximum predictive power out-of-sample.

Feature selection must not be confused with the data preprocessing steps during data preparation. Good feature selection requires an understanding of the data and statistics, and comprehensive EDA must be performed to assist with this step. Data preprocessing needs clarification only from data administrators and basic intuition (e.g., salary vs. income) during data preparation.

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Feature selection on structured data is a methodical and iterative process. Statistical measures can be used to assign a score gauging the importance of each feature. The features can then be ranked using this score and either retained or eliminated from the dataset. The statistical methods utilized for this task are usually univariate and consider each feature independently or with regard to the target variable. Methods include chi-square test, correlation coefficients, and information-gain measures (i.e., *R*-squared values from regression analysis). All of these statistical methods can be combined in a manner that uses each method individually on each feature, automatically performing backward and forward passes over features to improve feature selection. Prebuilt feature selection functions are available in popular programming languages used to build and train ML models.

Dimensionality reduction assists in identifying the features in the data that account for the greatest variance between observations and allows for the processing of a reduced volume of data. Dimensionality reduction may be implemented to reduce a large number of features, which helps reduce the memory needed and speed up learning algorithms. Feature selection is different from dimensionality reduction, but both methods seek to reduce the number of features in the dataset. The dimensionality reduction method creates new combinations of features that are uncorrelated, whereas feature selection includes and excludes features present in the data without altering them.

Feature Engineering

After the appropriate features are selected, feature engineering helps further optimize and improve the features. The success of ML model training depends on how well the data are presented to the model. The feature engineering process attempts to produce good features that describe the structures inherent in the dataset. This process depends on the context of the project, domain of the data, and nature of the problem. Structured data are likely to contain quantities, which can be engineered to better present relevant patterns in the dataset. This action involves engineering an existing feature into a new feature or decomposing it into multiple features.

For continuous data, a new feature may be created—for example, by taking the logarithm of the product of two or more features. As another example, when considering a salary or income feature, it may be important to recognize that different salary brackets impose a different taxation rate. Domain knowledge can be used to decompose an income feature into different tax brackets, resulting in a new feature: "income_above_100k," with possible values 0 and 1. The value 1 under the new feature captures the fact that a subject has an annual salary of more than $100,000. By grouping subjects into income categories, assumptions about income tax can be made and utilized in a model that uses the income tax implications of higher and lower salaries to make financial predictions.

For categorical data, for example, a new feature can be a combination (e.g., sum or product) of two features or a decomposition of one feature into many. If a single categorical feature represents education level with five possible values—high school, associates, bachelor's, master's, and doctorate—then these values can be decomposed into five new features, one for each possible value (e.g., is_highSchool, is_doctorate) filled with 0s (for false) and 1s (for true). The process in which categorical variables are converted into binary form (0 or 1) for machine reading is called **one hot encoding**. It is one of the most common methods for handling categorical features in text data. When date-time is present in the data, such features as "second of the hour," "hour of the day," and "day of the date" can be engineered to capture critical information about temporal data attributes—which are important, for example, in modeling trading algorithms.

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Feature engineering techniques systemically alter, decompose, or combine existing features to produce more meaningful features. More meaningful features allow an ML model to train more swiftly and easily. Different feature engineering strategies can lead to the generation of dramatically different results from the same ML model. The impact of feature selection and engineering on ML training is discussed further in the next section.

UNSTRUCTURED DATA: TEXT EXPLORATION

6

- ☐ describe objectives, methods, and examples of data exploration
- ☐ describe methods for extracting, selecting and engineering features from textual data

Exploratory Data Analysis

Just like with structured data, it is important to gain insight into existing patterns in the unstructured data for further analysis. In this section, text data will be discussed. Text analytics has various applications. The most common applications are text classification, topic modeling, fraud detection, and sentiment analysis. Text classification uses supervised ML approaches to classify texts into different classes. Topic modeling uses unsupervised ML approaches to group the texts in the dataset into topic clusters. Sentiment analysis predicts sentiment (negative, neutral, or positive) of the texts in a dataset using both supervised and unsupervised approaches.

Various statistics are used to explore, summarize, and analyze text data. Text data include a collection of texts (also known as a corpus) that are sequences of tokens. It is useful to perform EDA of text data by computing on the tokens such basic text statistics as **term frequency (TF)**, the ratio of the number of times a given token occurs in all the texts in the dataset to the total number of tokens in the dataset (e.g., word associations, average word and sentence length, and word and syllable counts).

Text statistics reveal patterns in the co-occurrence of words. There are many applications of text analytics, and necessary text statistics vary according to the context of the application. Topic modeling is a text data application in which the words that are most informative are identified by calculating the TF of each word. For example, the word "soccer" can be informative for the topic "sports." The words with high TF values are eliminated as they are likely to be stop words or other common vocabulary words, making the resulting BOW compact and more likely to be relevant to topics within the texts. In sentiment analysis and text classification applications, the chi-square measure of word association can be useful for understanding the significant word appearances in negative and positive sentences in the text or in different documents. The chi-square measure is further explained under feature selection. Such EDA plays a vital role in executing the feature selection step.

Text statistics can be visually comprehended by using the same methods as explained in the structured data section. For example, bar charts can be used to show word counts or frequency. Words clouds are common visualizations when working with text data as they can be made to visualize the most informative words and their TF values. The most commonly occurring words in the dataset can be shown by varying font size, and color is used to add more dimensions, such as frequency and length of words. Exhibit 17 shows a word cloud constructed from a sample dataset

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of generic financial news wires after text processing. Word cloud building functions and packages are available in several popular programming languages. A detailed demonstration of text data EDA will be presented in Section 7, where we work with actual text data in a financial forecasting project.

Exhibit 17: Word Cloud of Generic Financial Newsfeed Data Sample

Feature Selection

For text data, feature selection involves selecting a subset of the terms or tokens occurring in the dataset. The tokens serve as features for ML model training. Feature selection in text data effectively decreases the size of the vocabulary or BOW. This helps the ML model be more efficient and less complex. Another benefit is to eliminate noisy features from the dataset. Noisy features are tokens that do not contribute to ML model training and actually might detract from the ML model accuracy.

Noisy features are both the most frequent and most sparse (or rare) tokens in the dataset. On one end, noisy features can be stop words that are typically present frequently in all the texts across the dataset. On the other end, noisy features can be sparse terms that are present in only a few text cases. Text classification involves dividing text documents into assigned classes (a class is a category; examples include "relevant" and "irrelevant" text documents or "bearish" and "bullish" sentences). The *frequent* tokens strain the ML model to choose a decision boundary among the texts as the terms are present across all the texts, an example of model *underfitting*. The *rare* tokens mislead the ML model into classifying texts containing the rare terms into a specific class, an example of model *overfitting*. Identifying and removing noise features is very critical for text classification applications. The general feature selection methods in text data are as follows:

1. *Frequency* measures can be used for vocabulary pruning to remove noise features by filtering the tokens with very high and low TF values across all the texts. **Document frequency (DF)** is another frequency measure that helps to discard the noise features that carry no specific information about the text class and are present across all texts. The DF of a token is defined

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as the number of documents (texts) that contain the respective token divided by the total number of documents. It is the simplest feature selection method and often performs well when many thousands of tokens are present.

2. *Chi-square* test can be useful for feature selection in text data. The chi-square test is applied to test the independence of two events: occurrence of the token and occurrence of the class. The test ranks the tokens by their usefulness to each class in text classification problems. Tokens with the highest chi-square test statistic values occur more frequently in texts associated with a particular class and therefore can be selected for use as features for ML model training due to higher discriminatory potential.
3. *Mutual information* (MI) measures how much information is contributed by a token to a class of texts. The **mutual information** value will be equal to 0 if the token's distribution in all text classes is the same. The MI value approaches 1 as the token in any one class tends to occur more often in only that particular class of text. Exhibit 18 shows a simple depiction of some tokens with high MI scores for their corresponding text classes. Note how the tokens (or features) with the highest MI values narrowly relate to their corresponding text class name.

Exhibit 18: Tokens with Mutual Information (MI) Values for Two Given Text Classes

Text Classes: Sports or Politics			
Sports		**Politics**	
Token	**MI Value**	**Token**	**MI Value**
soccer	0.0781	election	0.0612
cup	0.0525	president	0.0511
match	0.0456	polls	0.0341
play	0.0387	vote	0.0288
game	0.0299	party	0.0202
team	0.0265	candidate	0.0201
win	0.0189	campaign	0.0201

Feature Engineering

As with structured data, feature engineering can greatly improve ML model training and remains a combination of art and science. The following are some techniques for feature engineering, which may overlap with text processing techniques.

1. *Numbers*: In text processing, numbers are converted into a token, such as "/number/." However, numbers can be of different lengths of digits representing different kinds of numbers, so it may be useful to convert different numbers into different tokens. For example, numbers with four digits may indicate years, and numbers with many digits could be an identification number. Four-digit numbers can be replaced with "/number4/," 10-digit numbers with "/number10/," and so forth.

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2. *N-grams*: Multi-word patterns that are particularly discriminative can be identified and their connection kept intact. For example, "market" is a common word that can be indicative of many subjects or classes; the words "stock market" are used in a particular context and may be helpful to distinguish general texts from finance-related texts. Here, a bigram would be useful as it treats the two adjacent words as a single token (e.g., stock_market).
3. *Name entity recognition (NER)*: NER is an extensive procedure available as a library or package in many programming languages. The **name entity recognition** algorithm analyzes the individual tokens and their surrounding semantics while referring to its dictionary to tag an object class to the token. Exhibit 19 shows the NER tags of the text "*CFA Institute was formed in 1947 and is headquartered in Virginia*." Additional object classes are, for example, MONEY, TIME, and PERCENT, which are not present in the example text. The NER tags, when applicable, can be used as features for ML model training for better model performance. NER tags can also help identify critical tokens on which such operations as lowercasing and stemming then can be avoided (e.g., Institute here refers to an organization rather than a verb). Such techniques make the features more discriminative.

Exhibit 19: Name Entity Recognition and Parts of Speech (POS) on Example Text

Token	NER Tag	POS Tag	POS Description
CFA	ORGANIZATION	NNP	Proper noun
Institute	ORGANIZATION	NNP	Proper noun
was		VBD	Verb, past tense
formed		VBN	Verb, past participle
in		IN	Preposition
1947	DATE	CD	Cardinal number
and		CC	Coordinating conjunction
is		VBZ	Verb, 3rd person singular present
headquartered		VBN	Verb, past participle
in		IN	Preposition
Virginia	LOCATION	NNP	Proper noun

4. *Parts of speech (POS)*: Similar to NER, **parts of speech** uses language structure and dictionaries to tag every token in the text with a corresponding part of speech. Some common POS tags are noun, verb, adjective, and proper noun. Exhibit 19 shows the POS tags and descriptions of tags for the example text. POS tags can be used as features for ML model training and to identify the number of tokens that belong to each POS tag. If a given text contains many proper nouns, it means that it may be related to people and organizations and may be a business topic. POS tags can be useful for separating verbs and nouns for text analytics. For example, the word "market" can be a verb when used as "to market …" or noun when used as "in the market." Differentiating such tokens can help further clarify the meaning of the text. The use of "market" as a verb could indicate that the text relates to the topic of marketing and might discuss marketing a product or service. The use of "market" as a noun could suggest that the text relates to a physical or stock market and might discuss stock trading. Also for POS tagging,

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such compound nouns as "CFA Institute" can be treated as a single token. POS tagging can be performed using libraries or packages in programming languages.

In addition, many more creative techniques convey text information in a structured way to the ML training process. The goal of feature engineering is to maintain the semantic essence of the text while simplifying and converting it into structured data for ML.

EXAMPLE 4

Data Exploration

Paul Wang's analytics team at LendALot Corporation has completed its initial data preparation and wrangling related to their creditworthiness classification ML model building efforts. As a next step, Wang has asked one of the team members, Eric Kim, to examine the available structured data sources to see what types of exploratory data analysis might make sense. Kim has been tasked with reporting to the team on high-level patterns and trends in the data and which variables seem interesting. Greater situational awareness about the data can inform the team's decisions regarding model training and whether (and how) to incorporate textual big data in conjunction with the structured data inputs. Use the following sample of columns and rows Kim pulled for manual examination to answer the next questions.

1	ID	Loan Outcome	Income (USD)	Loan Amount (USD)	Credit Score	Loan Type	Free Responses to "Explain Credit Score" (excerpts from full text)
2	1	No Default	34,000	10,000	685	Mortgage	I am embarrassed that my score is below 700, but it was due to mitigating circumstances. I have developed a plan to improve my score.
3	2	No Default	63,050	49,000	770	Student Loan	I have a good credit score and am constantly looking to further improve it…
4	3	Defaulted	20,565	35,000	730	Student Loan	I think I have great credit. I don't think there are any issues. Having to provide a written response to these questions is kind of annoying…
5	4	No Default	50,021	10,000	664	Mortgage	I have a decent credit score. I regret not being as responsible in the past but feel I have worked hard to improve my score recently…
6	5	Defaulted	100,350	129,000	705	Car Loan	Honestly, my score probably would have been higher if I had worked harder. But it is probably good enough…
7	6	No Default	800,000	300,000	800	Boat Loan	I have worked hard to maintain a good credit rating. I am very responsible. I maintain a payment schedule and always stick to the payment plan…

1. Evaluate whether data visualization techniques, such as histograms, box plots, and scatterplots, could be relevant to exploratory data analysis.

Solution:

The data provided include structured features (ID, Loan Outcome, Income, Loan Amount, Credit Score) and unstructured data. Histograms, box plots, and scatterplots are relevant visualization methods for structured data features. Histograms and box plots could be used by Kim to see how income,

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loan amount, and credit score are distributed. Moreover, these visualizations can be performed across all historical borrowing instances in the dataset as well as within the sets of defaulted loans versus non-defaulted loans. Scatterplots of income versus loan amount, income versus credit score, and loan amount versus credit score, both overall and within defaulted and non-defaulted datasets, can shed light on relationships between potentially important continuous variables.

2. State one visualization technique that could be used in relation to the free responses.

Solution:

For the text in the free response field, word clouds offer an appropriate starting point for exploratory analysis. A word cloud can enable a quick glimpse into the most frequently occurring words (i.e., term frequency). While some obvious words (e.g., "credit" and "score") may be valuable, other frequently occurring words (e.g., "worked," "hard," "probably," "embarrassed," "regret," "good," "decent," and "great") might have potential use for creditworthiness prediction.

3. Describe how ranking methods can be used to select potentially interesting features to report back to the team.

Solution:

Kim can use feature selection methods to rank all features. Since the target variable of interest (loan outcome) is discrete in this case, such techniques as chi-square and information gain would be well suited. These are univariate techniques that can score feature variables individually. In addition to the structured features, these univariate ranking methods can also be applied to word count-related features, such as term frequency and document frequency, that are derived from the text using frequently occurring words. Such frequently occurring words (e.g., "worked" and "hard") can be identified from the word cloud.

4. State an example of a bigram from the free response texts that could be used to discriminate among loan outcomes.

Solution:

The bigrams "credit_score" and "worked_hard" from the text in the free response section may have potential to discriminate among loan outcomes.

EXAMPLE 5

Textual Feature Representations for ML Model Building

Having completed their exploration of the data, Paul Wang's analytics team at LendALot Corporation recognizes the importance of incorporating features derived from text data in their ML models for classifying creditworthiness. Wang has asked his colleagues, Lynn Lee and Eric Kim, to propose textual feature representations that might be well suited to constructing features for their task. As a starting point, Lee and Kim review the following sample of data:

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1	ID	Loan Outcome	Income (USD)	Loan Amount (USD)	Credit Score	Loan Type	Free Responses to "Explain Credit Score" (excerpts from full text)
2	1	No Default	34,000	10,000	685	Mortgage	I am embarrassed that my score is below 700, but it was due to mitigating circumstances. I have developed a plan to improve my score.
3	2	No Default	63,050	49,000	770	Student Loan	I have a good credit score and am constantly looking to further improve it…
4	3	Defaulted	20,565	35,000	730	Student Loan	I think I have great credit. I don't think there are any issues. Having to provide a written response to these questions is kind of annoying…
5	4	No Default	50,021	10,000	664	Mortgage	I have a decent credit score. I regret not being as responsible in the past but feel I have worked hard to improve my score recently…
6	5	Defaulted	100,350	129,000	705	Car Loan	Honestly, my score probably would have been higher if I had worked harder. But it is probably good enough…
7	6	No Default	800,000	300,000	800	Boat Loan	I have worked hard to maintain a good credit rating. I am very responsible. I maintain a payment schedule and always stick to the payment plan…

Based on the information given, address the following questions.

1. Describe three textual feature representations that Lee and Kim should consider for their text data.

Solution:

Lee and Kim should consider bag-of-words (BOW), n-grams, and parts-of-speech (POS) as key textual feature representations for their text data. Conversely, name entity recognition (NER) might not be as applicable in this context because the data on prospective borrowers does not include any explicit references to people, locations, dates, or organizations.

2. Describe a rationale for adopting each of the three textual feature representations identified in Question 1.

Solution:

All three textual feature representations have the potential to add value.

Bag-of-words (BOW) is typically applicable in most contexts involving text features derived from languages where token boundaries are explicitly present (e.g., English) or can be inferred through processing (e.g., a different language, such as Spanish). BOW is generally the best starting point for most projects exploring text feature representations.

N-grams, representations of word or token sequences, are also applicable. N-grams can offer invaluable contextual information that can complement and enrich a BOW. In this specific credit-worthiness context, we examine the BOW token "worked." It appears three times (rows 5–7), twice in no-default loan texts and once in a defaulted loan text. This finding suggests that "worked" is being used to refer to the borrower's work ethic and may be a good predictor of credit worthiness. Digging deeper and looking at several trigrams (i.e., three-token sequences) involving "worked," we see that "have_worked_hard" appears in the two no-default loan related texts (referring to borrower accomplishments and plans) and "had_worked_harder" appears in

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the defaulted loan text (referring to what could have been done). This example illustrates how n-grams can provide richer contextualization capabilities for the creditworthiness prediction ML models.
Parts-of-speech tags can add value because they identify the composition of the texts. For example, POS provides information on whether the prospective borrowers are including many action words (verbs) or descriptors (adjectives) and whether this is being done differently in instances of no-default versus instances of defaulted loans.

7 MODEL TRAINING, STRUCTURED VS. UNSTRUCTURED DATA, AND METHOD SELECTION

☐ describe objectives, steps, and techniques in model training

Machine learning model training is a systematic, iterative, and recursive process. The number of iterations required to reach optimum results depends on:

- the nature of the problem and input data and
- the level of model performance needed for practical application.

Machine learning models combine multiple principles and operations to provide predictions. As seen in the last two sections, typical ML model building requires data preparation and wrangling (cleansing and preprocessing) and data exploration (exploratory data analysis as well as feature selection and engineering). In addition, domain knowledge related to the nature of the data is required for good model building and training. For instance, knowledge of investment management and securities trading is important when using financial data to train a model for predicting costs of trading stocks. It is crucial for ML engineers and domain experts to work together in building and training robust ML models.

The three tasks of ML model training are method selection, performance evaluation, and tuning. Exhibit 20 outlines model training and its three component tasks. Method selection is the art and science of deciding which ML method(s) to incorporate and is guided by such considerations as the classification task, type of data, and size of data. Performance evaluation entails using an array of complementary techniques and measures to quantify and understand a model's performance. Tuning is the process of undertaking decisions and actions to improve model performance. These steps may be repeated multiple times until the desired level of ML model performance is attained. Although no standard rulebook for training an ML model exists, having a fundamental understanding of domain-specific training data and ML algorithm principles plays a vital role in good model training.

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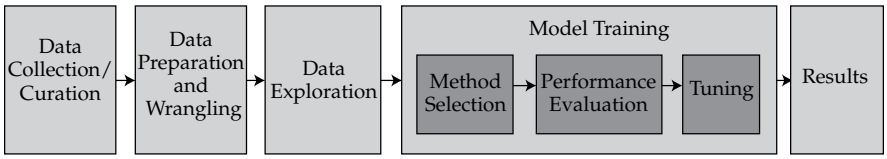
Exhibit 20: Model Training Stage



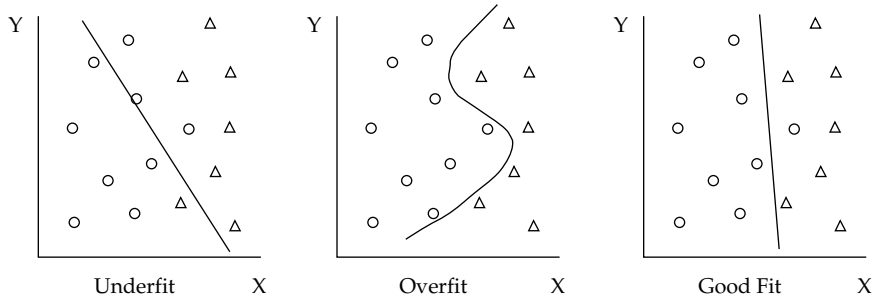
Before training a model, it is important to state the problem, define objectives, identify useful data points, and conceptualize the model. Conceptualization is like a blueprint on a drawing board, a modifiable plan that is necessary to initiate the model training process. Because modeling is an iterative process, many changes and refinements will be made to the model plan as the process evolves.

Structured and Unstructured Data

The ML model training process for structured and unstructured data is typically the same. Most ML models are intended to train on structured data, so unstructured data in the data preparation stage are processed and organized into a structured format. The systematic processing of unstructured text data so that they can be structured in the form of a data matrix has been previously covered. Similarly, other forms of unstructured data can also be prepared and formed into data matrixes or tables for ML training.

The fundamental idea of ML model training is fitting a system of rules on a training dataset to reveal a pattern in the data. In other words, fitting describes the degree to which (or how well) an ML model can be generalized to new data. A good model fit results in good model performance and can be validated using new data outside of the training dataset (i.e., out-of-sample). Exhibit 21 shows model decision boundaries in three possible model fitting scenarios for a classification task comprising two different classes of data (i.e., circles and triangles). The model on the left is underfit; it does not fit the training data well enough since it results in four misclassification errors (three circles and one triangle). Although the center model that generates the "S"-shaped line has the best accuracy (no errors) on the training data, it is overfit (i.e., fits the training data too well) and thus unlikely to perform well on future test cases. The model on the right (with one classification error, a circle) is a model with good fit (i.e., it fits the training data well but not so well that it cannot be generalized to out-of-sample data).

Exhibit 21: Model Fitting Scenarios: Underfit, Overfit, and Good Fit



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Model fitting errors are caused by several factors—the main ones being dataset size and number of features in the dataset.

- *Dataset Size*: Small datasets can lead to underfitting of the model since small datasets often are not sufficient to expose patterns in the data. Restricted by a small dataset, an ML model may not recognize important patterns.
- *Number of Features*: A dataset with a small number of features can lead to underfitting, and a dataset with a large number of features can lead to overfitting. As with small dataset size, a small number of features may not carry all the characteristics that explain relationships between the target variable and the features. Conversely, a large number of features can complicate the model and potentially distort patterns in the data due to low degrees of freedom, causing overfitting. Therefore, appropriate feature selection using the types of techniques described earlier (e.g., chi-square, mutual information) is a key factor in minimizing such model overfitting.

Feature engineering tends to prevent underfitting in the training of the model. New features, when engineered properly, can elevate the underlying data points that better explain the interactions of features. Thus, feature engineering can be critical to overcome underfitting. Method-related factors that affect model fitting are explained shortly under tuning.

Method Selection

ML model training is a craft (part art and part science); it has no strict guidelines. Selecting and applying a method or an algorithm is the first step of the training process. Method selection is governed by the following factors:

1. *Supervised or unsupervised learning*. The data for training and testing supervised ML models contain **ground truth**, the known outcome (i.e., target variable) of each observation in these datasets. Unsupervised ML modeling is relatively challenging because of the absence of ground truth (i.e., no target variable). Supervised models bring a structure that may or may not be supported by the data. Unsupervised models bring no structure beyond that which arises from the given data. For supervised learning (with labeled training data), typical methods of choice are regression, ensemble trees, support vector machines (SVMs), and neural networks (NNs). Supervised learning would be used, for example, for default prediction based on high-yield corporate bond issuer data. For unsupervised learning, common methods are dimensionality reduction, clustering, and anomaly detection. Unsupervised learning, for example, would be used for clustering financial institutions into different groups based on their financial attributes.
2. *Type of data*. For numerical data (e.g., predicting stock prices using historical stock market values), classification and regression tree (CART) methods may be suitable. For text data (for example, predicting the topic of a financial news article by reading the headline of the article), such methods as generalized linear models (GLMs) and SVMs are commonly used. For image data (e.g., identifying objects in a satellite image, such as tanker ships moving in and out of port), NNs and deep learning methods tend to perform better than others. For speech data (e.g., predicting financial sentiment from quarterly earnings' conference call recordings), deep learning methods can offer promising results.
3. *Size of data.* A typical dataset has two basic characteristics: number of instances (i.e., observations) and number of features. The combination of these two characteristics can govern which method is most suitable for model training. For instance, SVMs have been found to work well on

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"wider" datasets with 10,000 to 100,000 features and with fewer instances. Conversely, NNs often work better on "longer" datasets, where the number of instances is much larger than the number of features.

Once a method is selected, certain method-related decisions (e.g., on hyperparameters) need to be made. These decisions include the number of hidden layers in a neural network and the number of trees in ensemble methods (discussed later in the sub-section on tuning). In practice, datasets can be a combination of numerical and text data. To deal with mixed data, the results from more than one method can be combined. Sometimes, the predictions from one method can be used as predictors (features) by another. For example, unstructured financial text data can be used with logistic regression to classify stock sentiment as either positive or negative. Then, this sentiment classification cam be used as a predictor in a larger model, say CART, that also uses structured financial data as predictors for the purpose of stock selection. Finally, more than one method can be used and the results combined with quantitative or subjective weighing to exploit the advantages of each method.

Before model training begins, in the case of supervised learning the master dataset is split into three subsets used for model training and testing purposes. The first subset, a training set used to train the model, should constitute approximately 60% of the master dataset. The second subset, a cross-validation set (or validation set) used to tune and validate the model, should constitute approximately 20% of the master dataset. The third subset is a test set for testing the model and uses the remaining data. The data are split using a random sampling technique, such as the k-fold method. A commonly recommended split ratio is 60:20:20, as detailed above; however, the split percentages can vary. For unsupervised learning, no splitting is needed due to the absence of labeled training data.

Class imbalance, where the number of instances for a particular class is significantly larger than for other classes, may be a problem for data used in supervised learning because the ML classification method's objective is to train a high-accuracy model. In a high-yield bond default prediction example, say for corporate issuers in the BB+/Ba1 to B+/B1 credit quality range, issuers who defaulted (positive or "1" class) would be very few compared to issuers who did not default (negative or "0" class). Hence, on such training data, a naive model that simply assumes no corporate issuer will default may achieve good accuracy—albeit with all default cases misclassified. Balancing the training data can help alleviate such problems. In cases of unbalanced data, the "0" class (majority class) can be randomly undersampled or the "1" class (minority class) randomly oversampled. The random sampling can be done with or without replacement because they both work the same in general probability theory. Exhibit 22 depicts the idea of undersampling of the majority class and oversampling of the minority class. In practice, the choice of whether to undersample or oversample depends on the specific problem context. Advanced techniques can also reproduce synthetic observations from the existing data, and the new observations can be added to the dataset to balance the minority class.

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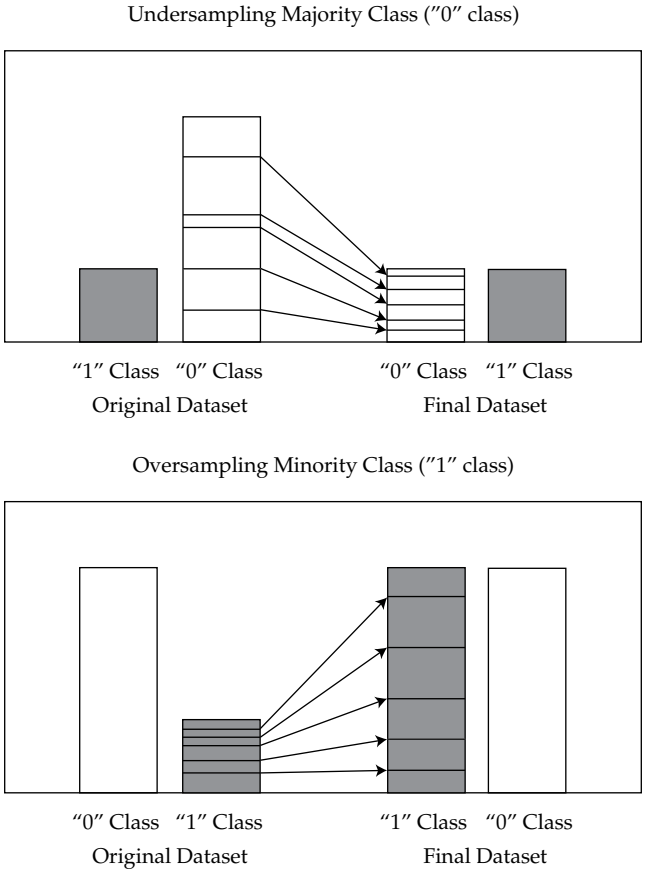
Exhibit 22: Undersampling and Oversampling



8 PERFORMANCE EVALUATION

☐ describe objectives, steps, and techniques in model training

It is important to measure the model training performance or goodness of fit for validation of the model. We shall cover several techniques to measure model performance that are well suited specifically for binary classification models.

1. *Error analysis.* For classification problems, error analysis involves computing four basic evaluation metrics: true positive (TP), false positive (FP), true negative (TN), and false negative (FN) metrics. FP is also called a Type I error, and FN is also called a Type II error. Exhibit 23 shows a **confusion matrix**, a grid that is used to summarize values of these four metrics.

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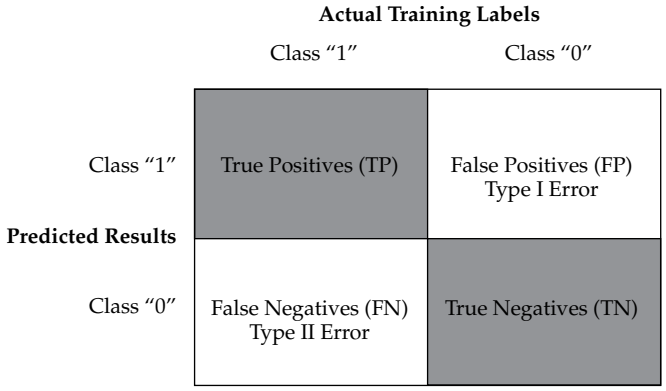
Exhibit 23: Confusion Matrix for Error Analysis



Additional metrics, such as precision and recall, can be computed. Assume in the following explanation that Class "0" is "not defective" and Class "1" is "defective." **Precision** is the ratio of correctly predicted positive classes to all predicted positive classes. Precision is useful in situations where the cost of FP, or Type I error, is high—or example, when an expensive product fails quality inspection (predicted Class "1") and is scrapped, but it is actually perfectly good (actual Class "0"). **Recall** (also known as *sensitivity*) is the ratio of correctly predicted positive classes to all actual positive classes. Recall is useful in situations where the cost of FN or Type II error is high—for example, when an expensive product passes quality inspection (predicted Class "0") and is sent to the valued customer, but it is actually quite defective (actual Class "1"). The formulas for precision and recall are:

$$\text{Precision (P)} = TP/(TP + FP). \tag{3}$$

$$\text{Recall (R)} = TP/(TP + FN). \tag{4}$$

Trading off precision and recall is subject to business decisions and model application. Therefore, additional evaluation metrics that provide the overall performance of the model are generally used. The two overall performance metrics are accuracy and F1 score. **Accuracy** is the percentage of correctly predicted classes out of total predictions. **F1 score** is the harmonic mean of precision and recall. F1 score is more appropriate (than accuracy) when unequal class distribution is in the dataset and it is necessary to measure the equilibrium of precision and recall. High scores on both of these metrics suggest good model performance. The formulas for accuracy and F1 score are as follows:

$$\text{Accuracy} = (TP + TN)/(TP + FP + TN + FN). \tag{5}$$

$$\text{F1 score} = (2 * P * R)/(P + R). \tag{6}$$

Exhibit 24 illustrates computations of model evaluation metrics and performance scores on a sample dataset.

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Exhibit 24: Performance Metrics and Scores Computation

Sample Dataset with Classification Results

Observation	Actual Training Labels	Predicted Results	Classification
1	1	1	TP
2	0	0	TN
3	1	1	TP
4	1	0	FN
5	1	1	TP
6	1	0	FN
7	0	0	TN
8	0	0	TN
9	0	0	TN
10	0	1	FP

Confusion Matrix

		Actual Training Labels	
		Class "1"	Class "0"
Predicted Results	Class "1"	3 (TP)	1 (FP)
	Class "0"	2 (FN)	4 (TN)

Performance Metrics

TP = 3, FP = 1, FN = 2, TN = 4

P = 3 / (3+1) = 0.75

R = 3 / (3+2) = 0.60

F1 Score = (2 × 0.75 × 0.60) / (0.75 + 0.60) = 0.67

Accuracy = (3 + 4) / (3 + 1 + 4 + 2) = 0.70

In Exhibit 24, if all "1" classes were predicted correctly (no FPs), the precision would have been equal to 1. If all "0" classes were predicted correctly (no FNs), the recall would have been equal to 1. Thus, the resulting F1 score would have been equal to 1. The precision of 0.75 and recall of 0.60 indicate that the model is better at minimizing FPs than FNs. To find the equilibrium between precision and recall, F1 score is calculated, which is equal to 0.67. The F1 score is closer to the smaller value among both precision and recall, giving the model a more appropriate score rather than just an arithmetic mean. Accuracy, the percentage of correct predictions (for both classes) made by the model, is equal to 0.70. Accuracy would be equal to 1 if all predictions were correct. As the number of "1" and "0" classes is equal in the dataset (i.e., a balanced dataset), accuracy can be considered an appropriate performance measure in this case. If the number of classes in a dataset is unequal; however, then F1 score should be used as the overall performance measure for the model.

2. *Receiver Operating Characteristic (ROC).* This technique for assessing model performance involves the plot of a curve showing the trade-off between the false positive rate (x-axis) and true positive rate (y-axis) for various cutoff points—for example, for the predicted probability (p) in a logistic regression. The formulas for false positive rate and true positive rate (note that true positive rate is the same as recall) are:

False positive rate (FPR) = FP/(TN + FP) and (7)

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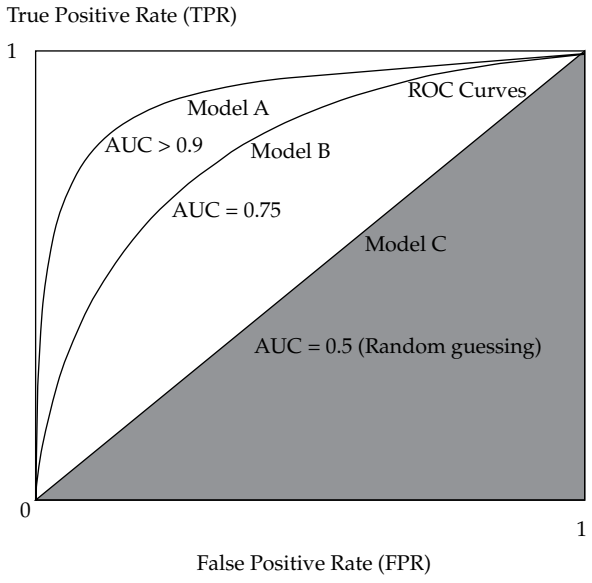
True positive rate (TPR) = TP/(TP + FN). (8)

If p from a logistic regression model for a given observation is greater than the cutoff point (or threshold), then the observation is classified as class = 1. Otherwise, the observation will be classified as class = 0.

The shape of the ROC curve provides insight into the model's performance. A more convex curve indicates better model performance. Area under the curve (AUC) is the metric that measures the area under the ROC curve. An AUC close to 1.0 indicates near perfect prediction, while an AUC of 0.5 signifies random guessing. Exhibit 25 displays three ROC curves and indicates their respective AUC values. It is clear from observing the shapes of the ROC curves and their AUCs that Model A—with the most convex ROC curve with AUC of more than 0.9 (or 90%)—is the best performing among the three models.

Exhibit 25: ROC Curves and AUCs



3. *Root Mean Squared Error (RMSE).* This measure is appropriate for continuous data prediction and is mostly used for regression methods. It is a single metric that captures all the prediction errors in the data (n). The root mean squared error is computed by finding the square root of the mean of the squared differences between the actual values and the model's predicted values (error). A small RMSE indicates potentially better model performance. The formula for RMSE is:

$$\text{RMSE} = \sqrt{\sum_{i=1}^{n} \frac{\left(\text{Predicted}_i - \text{Actual}_i\right)^2}{n}}. \quad (9)$$

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9 TUNING

☐ describe objectives, steps, and techniques in model training

Once the model is evaluated, certain decisions and actions must be taken based on the findings to improve the performance of the model. If the prediction error on the training set is high, the model is underfitting. If the prediction error on the cross-validation (CV) set is significantly higher than on the training set, the model is overfitting. Model fitting has two types of error: bias and variance. Bias error is associated with underfitting, and variance error is associated with overfitting. Bias error is high when a model is overly simplified and does not sufficiently learn from the patterns in the training data. Variance error is high when the model is overly complicated and memorizes the training data so much that it will likely perform poorly on new data. It is not possible to completely eliminate both types of errors. However, both errors can be minimized so the total aggregate error (bias error + variance error) is at a minimum. The bias–variance trade-off is critical to finding an optimum balance where a model neither underfits nor overfits.

1. *Parameters* are critical for a model and are dependent on the training data. Parameters are learned from the training data as part of the training process by an optimization technique. Examples of parameters include coefficients in regression, weights in NN, and support vectors in SVM.
2. *Hyperparameters* are used for estimating model parameters and are not dependent on the training data. Examples of hyperparameters include the regularization term (λ) in supervised models, activation function and number of hidden layers in NN, number of trees and tree depth in ensemble methods, *k* in k-nearest neighbor classification and k-means clustering, and p-threshold in logistic regression. Hyperparameters are manually set and tuned.

For example, if a researcher is using a logistic regression model to classify sentences from financial statements into positive or negative stock sentiment, the initial cutoff point for the trained model might be a p-threshold of 0.50 (50%). Therefore, any sentence for which the model produces a probability >50% is classified as having positive sentiment. The researcher can create a confusion matrix from the classification results (of running the CV dataset) to determine such model performance metrics as accuracy and F1 score. Next, the researcher can vary the logistic regression's p-threshold—say to 0.55 (55%), 0.60 (60%), or even 0.65 (65%)—and then re-run the CV set, create new confusion matrixes from the new classification results, and compare accuracy and F1 scores. Ultimately, the researcher would select the logistic regression model with a p-threshold value that produces classification results generating the highest accuracy and F1 scores. Note that the process just outlined will be demonstrated in Section 7.

There is no general formula to estimate hyperparameters. Thus, tuning heuristics and such techniques as grid search are used to obtain the optimum values of hyperparameters. **Grid search** is a method of systematically training an ML model by using various combinations of hyperparameter values, cross validating each model, and determining which combination of hyperparameter values ensures the best model performance. The model is trained using different combinations of hyperparameter values until the optimum set of values are found. Optimum values must result in similar performance of the model on training and CV datasets, meaning that the training error and CV error are close. This ensures that the model can be generalized to test data or to new data and thus is less likely to overfit. The plot of training errors for

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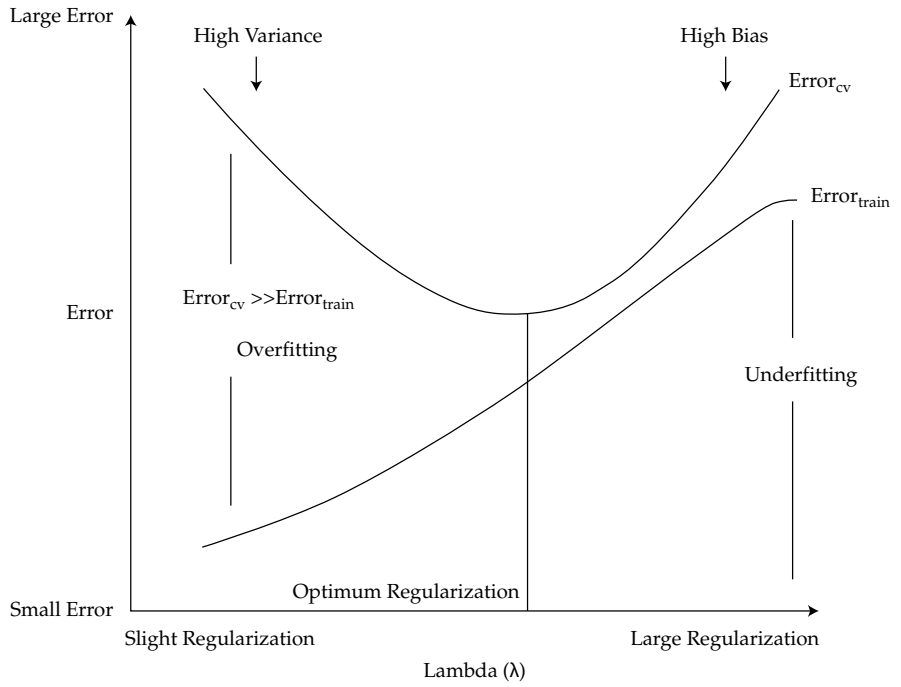
each value of a hyperparameter (i.e., changing model complexity) is called a fitting curve. Fitting curves provide visual insight on the model's performance (for the given hyperparameter and level of model complexity) on the training and CV datasets and are visually helpful to tune hyperparameters. Exhibit 26 shows the bias–variance error trade-off by plotting a generic fitting curve for a regularization hyperparameter (λ).

Exhibit 26: Fitting Curve for Regularization Hyperparameter (λ)



Slight regularization lightly penalizes model complexity, thereby allowing most or all of the features to be included in the model and thus potentially enabling the model to "memorize" the data. Typically with no or slight regularization, the prediction error on the training dataset is small while the prediction error on the CV dataset is significantly larger. This difference in error is variance. High variance error, which typically results from too many features and model complexity, results in model overfitting. When high variance error and low bias error exist, the model performs well on the training dataset but generates many FP and FN errors on the CV dataset; in other words, the model is overfitted and does not generalize to new data well.

Large regularization excessively penalizes model complexity, thereby allowing too few of the features to be included in the model and causing the model to learn less from the data. The model may lack the necessary predictor variables and complexity needed to discern underlying patterns in the data. Typically with large regularization, the prediction errors on the training and CV datasets are both large. Large prediction errors on the training dataset indicate high bias, and high bias error results from model underfitting. When high bias error exists, the model does not perform well on either training or CV datasets because it is typically lacking important predictor variables.

Optimum regularization minimizes both variance and bias errors in a balanced fashion. It penalizes model complexity just enough so that only the most important features are included in the model. This process prevents the model from memorizing the data while enabling the model to learn enough from the data to distinguish

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important patterns. This results in prediction errors in both training and CV datasets that are similar and also minimal. The range of optimum regularization values can be found heuristically using such techniques as grid search.

If high bias or variance exists after the tuning of hyperparameters, either a larger number of training examples (instances) may be needed or the number of features included in the model may need to be decreased (in the case of high variance) or increased (in the case of high bias). The model then needs to be re-trained and re-tuned using the new training dataset. In the case of a complex model, where a large model is comprised of sub-model(s), ceiling analysis can be performed. **Ceiling analysis** is a systematic process of evaluating different components in the pipeline of model building. It helps to understand what part of the pipeline can potentially improve in performance by further tuning. For example, a stock market prediction model needs historical data from the stock market and perhaps news articles related to the stocks. The sub-model will extract relevant information from the news articles or classify the sentiment of the news articles. The results of the sub-model will feed into the larger model as features. Thus, the performance of the larger model depends on performance of the sub-model(s). Ceiling analysis can help determine which sub-model needs to be tuned to improve the overall accuracy of the larger model.

10 FINANCIAL FORECASTING PROJECT

- describe preparing, wrangling, and exploring text-based data for financial forecasting

Robo-readers are automated programs used to analyze large quantities of text, including news articles and social media. In the financial services space, robo-readers are being used by investors to examine how views expressed in text relate to future company performance. One important dimension that robo-readers look to analyze is sentiment polarity—which means how positive, negative, or neutral a particular phrase or statement is regarding a "target." For example, in the statement "XYZ Corporation is doing terrific things with its new product innovation," positive sentiment (i.e., the polarity) is being expressed regarding XYZ Corporation (i.e., the target of the sentiment). Such sentiment can provide invaluable predictive power, both alone and when coupled with structured financial data, for predicting stock price movements for individual firms and for portfolios of companies.

To provide a practical application, we use a financial forecasting project to examine how effectively sentiment—expressed in English news articles on LexisNexis (a searchable database of news articles) related to all companies listed on the NASDAQ OMX Helsinki (Finland)—can be classified. To accomplish this task, we followed the text ML model building steps presented in Sections 3 to 6 of this reading.

Text Curation, Preparation, and Wrangling

Text Curation

The text data used in this financial forecasting project are a collection of English language sentences from financial and economic news sources. The text data are acquired from the Financial Phrase Bank located on the website Researchgate.net.[2] The com-

2 https://www.researchgate.net/publication/251231364_FinancialPhraseBank-v10.

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pressed folder contains six text files. The first two files are license and readme files. The other four files contain the text data. The data are presented in a text document format (.txt), which can be opened and viewed using any text editor. Note that this is cross-sectional data (not time series data).

A total of 14,780 sentences are in the four files. The sentiment of each sentence has already been labeled with one of three sentiment classes: positive, neutral, or negative. The sentiment classes are provided from an investor's perspective and may be useful for predicting whether a sentence may have a corresponding positive, neutral, or negative influence on the respective company's stock price.

This project uses sentences from two of the text files (Sentences_AllAgree and Sentences_75Agree), labeled as either in the positive or negative sentiment class, for a total of 2,180 sentences. There are 1,457 positive sentiment class sentences and 723 negative sentiment class sentences. A supervised ML model is trained, validated, and tested using these data. The final ML model can be used to predict the sentiment classes of sentences present in similar financial news statements. Exhibit 27 shows a sample of 10 rows of raw text from the Sentences_AllAgree text file. Note the sentiment annotations at the end of each sentence with prefix character "@."

Exhibit 27: Ten Sample Sentences and Sentiment from Raw Text File (Sentences_AllAgree.txt)

```
Profit before taxes amounted to EUR 56.5 mn , down from EUR 232.9 mn a year ago .@negative
Profit before taxes decreased by 9 % to EUR 187.8 mn in the first nine months of 2008 , compared to EUR 207.1 mn a year earlier .@negative
Profit before taxes decreased to EUR 31.6 mn from EUR 50.0 mn the year before .@negative
Profit before taxes was EUR 4.0 mn , down from EUR 4.9 mn .@negative
The company 's profit before taxes fell to EUR 21.1 mn in the third quarter of 2008 , compared to EUR 35.8 mn in the corresponding period in 2007 .@negative
In August-October 2010 , the company 's result before taxes totalled EUR 9.6 mn , up from EUR 0.5 mn in the corresponding period in 2009 .@positive
Finnish Bore that is owned by the Rettig family has grown recently through the acquisition of smaller shipping companies .@positive
The plan is estimated to generate some EUR 5 million ( USD 6.5 m ) in cost savings on an annual basis .@positive
Finnish pharmaceuticals company Orion reports profit before taxes of EUR 70.0 mn in the third quarter of 2010 , up from EUR 54.9 mn in the corresponding period in 2009 .@positive
Finnish Sampo Bank , of Danish Danske Bank group , reports profit before taxes of EUR 152.3 mn in 2010 , up from EUR 32.7 mn in 2009 .@positive
```

Text Preparation (Cleansing)

The raw text data (i.e., sentences) are initially organized into a data table. The data table contains two columns: The first column (sentence) is for the text, and the second column (sentiment) is for the corresponding sentiment class. The separator character, which is "@" in this case, is used to split the data into text and sentiment class columns. A collection of text data in any form, including list, matrix, or data table forms, is called a **corpus**. Exhibit 28 shows a sample of 10 sentences from the data table corpus.

Exhibit 28: Ten Sample Rows of the Data Table (Corpus)

Sentence	Sentiment
Profit before taxes amounted to EUR 56.5 mn , down from EUR 232.9 mn a year ago .	negative
Profit before taxes decreased by 9 % to EUR 187.8 mn in the first nine months of 2008 , compared to EUR 207.1 mn a year earlier .	negative
Profit before taxes decreased to EUR 31.6 mn from EUR 50.0 mn the year before .	negative
Profit before taxes was EUR 4.0 mn , down from EUR 4.9 mn .	negative
The company 's profit before taxes fell to EUR 21.1 mn in the third quarter of 2008 , compared to EUR 35.8 mn in the corresponding period in 2007 .	negative
In August-October 2010 , the company 's result before taxes totalled EUR 9.6 mn , up from EUR 0.5 mn in the corresponding period in 2009 .	positive

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Sentence	Sentiment
Finnish Bore that is owned by the Rettig family has grown recently through the acquisition of smaller shipping companies .	positive
The plan is estimated to generate some EUR 5 million (USD 6.5 m) in cost savings on an annual basis .	positive
Finnish pharmaceuticals company Orion reports profit before taxes of EUR 70.0 mn in the third quarter of 2010 , up from EUR 54.9 mn in the corresponding period in 2009 .	positive
Finnish Sampo Bank , of Danish Danske Bank group , reports profit before taxes of EUR 152.3 mn in 2010 , up from EUR 32.7 mn in 2009 .	positive

The raw text contains punctuations, numbers, and white spaces that may not be necessary for model training. Text cleansing involves removing, or incorporating appropriate substitutions for, potentially extraneous information present in the text. Operations to remove html tags are unnecessary because none are present in the text

Punctuations: Before stripping out punctuations, percentage and dollar symbols are substituted with word annotations to retain their essence in the financial texts. Such word annotation substitutions convey that percentage and currency-related tokens were involved in the text. As the sentences have already been identified within and extracted from the source text, punctuation helpful for identifying discrete sentences—such as periods, semi-colons, and commas—are removed. Some special characters, such as "+" and "©," are also removed. It is a good practice to implement word annotation substitutions before removing the rest of the punctuations.

Numbers: Numerical values of numbers in the text have no significant utility for sentiment prediction in this project because sentiment primarily depends on the words in a sentence. Here is an example sentence: "*Ragutis, which is based in Lithuania's second-largest city, Kaunas, boosted its sales last year 22.3 percent to 36.4 million litas*." The word "boosted" implies that there was growth in sales, so analysis of this sentiment does not need to rely on interpretation of numerical text data. Sentiment analysis typically does not involve extracting, interpreting, and calculating relevant numbers but instead seeks to understand the context in which the numbers are used. Other commonly occurring numbers are dates and years, which are also not required to predict sentence sentiment. Thus, all numbers present in the text are removed for this financial sentiment project. However, prior to removing numbers, abbreviations representing orders of magnitude, such as million (commonly represented by "m," "mln," or "mn"), billion, or trillion, are replaced with the complete word. Retaining these orders of magnitude-identifying words in the text preserves the original text meaning and can be useful in predicting sentence sentiment.

Whitespaces: White spaces are present in the raw text. Additional white spaces occur after performing the above operations to remove extraneous characters. The white spaces must be removed to keep the text intact. Exhibit 29 shows the sample text after cleansing. The cleansed text is free of punctuations and numbers, with useful substitutions.

Exhibit 29: Ten Sample Rows After Cleansing Process

Sentence	Sentiment
Profit before taxes amounted to EUR million down from EUR million a year ago	negative
Profit before taxes decreased by percentSign to EUR million in the first nine months of compared to EUR million a year earlier	negative
Profit before taxes decreased to EUR million from EUR million the year before	negative
Profit before taxes was EUR million down from EUR million	negative

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Sentence	Sentiment
The companys profit before taxes fell to EUR million in the third quarter of compared to EUR million in the corresponding period in	negative
In August October the companys result before taxes totalled EUR million up from EUR million in the corresponding period in	positive
Finnish Bore that is owned by the Rettig family has grown recently through the acquisition of smaller shipping companies	positive
The plan is estimated to generate some EUR million USD million in cost savings on an annual basis	positive
Finnish pharmaceuticals company Orion reports profit before taxes of EUR million in the third quarter of up from EUR million in the corresponding period in	positive
Finnish Sampo Bank of Danish Danske Bank group reports profit before taxes of EUR million in up from EUR million in	positive

Text Wrangling (Preprocessing)

The cleansed text needs to be normalized using the following normalization procedures:

1. *Lowercasing* of all text to consolidate duplicate words (example, "THE," "The," and "the").
2. *Stop words* are not removed because some stop words (e.g., not, more, very, and few) carry significant meaning in the financial texts that is useful for sentiment prediction. Some stop words, such as articles (a, an, the), may be removed. Nevertheless, to avoid confusion no words are removed at this point. This issue will be revisited during the data exploration stage, which will carefully examine the text using frequency analysis and find custom stop words (common words) for these particular text data.
3. *Stemming*, the converting of inflected forms of a word into its base word (stem), is performed on the text as it is simple to perform and is appropriate for training an ML model for sentiment prediction.

White spaces are stripped after performing these operations. As part of text normalization, different currency abbreviations, such as EUR and USD, can be converted into a single token, such as "currencysign." As we are dealing with financial domain text, the earlier substitution of dollarsign can be replaced with currencysign as well. This step will remove tokens that are different but redundant in nature while maintaining their meaning. Through careful examination of the text and use of domain knowledge, similar substitutions of redundant tokens can be performed. Exhibit 30 shows how the sample text appears after normalization.

Exhibit 30: Ten Sample Rows After Normalization Process

Sentence	Sentiment
profit befor tax amount to currencysign million down from currencysign million a year ago	negative
profit befor tax decreas by percentsign to currencysign million in the first nine month of compar to currencysign million a year earlier	negative
profit before tax decreas to currencysign million from currencysign million the year befor	negative
profit befor tax was currencysign million down from currencysign million	negative
the compani profit befor tax fell to currencysign million in the third quarter of compar to currencysign million in the correspond period in	negative
in august octob the compani result befor tax total currencysign million up from currencysign million in the correspond period in	positive

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Sentence	Sentiment
finnish bore that is own by the rettig famili has grown recent through the acquisit of smaller shipping company	positive
the plan is estim to generat some currencysign million currencysign million in cost save on an annual basi	positive
finnish pharmaceut compani orion report profit befor tax of currencysign million in the third quarter of up from currencysign million in the correspond period in	positive
finnish sampo bank of danish danske bank group report profit befor tax of currencysign million in up from currencysign million in	positive

The normalized text is tokenized, resulting in 2,673 unique tokens. Altogether, these unique tokens comprise the bag-of-words (BOW) of the text corpus. Exhibit 31 shows a sample of 100 tokens from the BOW. This preliminary unigram BOW can be used to construct a document term matrix (DTM) for ML training.

Exhibit 31: One Hundred Sample Tokens from Preliminary Unigram BOW

"for"	"foundri"	"quarter"	"shop"	"net"	"share"	"to"
"currencysign"	"nokia"	"same"	"plan"	"year"	"sanyo"	"it"
"move"	"nokian"	"tax"	"earn"	"in"	"expect"	"by"
"percentsign"	"director"	"rose"	"dividned"	"total"	"megafon"	"talentum"
"report"	"as"	"chain"	"number"	"consolid"	"accord"	"compar"
"prior"	"last"	"machin"	"componenta"	"afx"	"doubl"	"higher"
"led"	"from"	"announc"	"a"	"with"	"while"	"g"
"handset"	"pre"	"fourth"	"loss"	"analyst"	"increas"	"said"
"board"	"oper"	"propos"	"repres"	"paid"	"finnish"	"base"
"user"	"retail"	"market"	"is"	"late"	"amount"	"estim"
"the"	"divis"	"of"	"helsinki"	"sale"	"close"	
"million"	"after"	"period"	"team"	"earlier"	"manufactur"	
"zero"	"tyre"	"profit"	"beat"	"third"	"dealer"	
"and"	"will"	"correspond"	"per"	"up"	"subscrib"	
"cloth"	"decemb"	"sepp"	"custom"	"reach"	"teliasonera"	

The final DTM for ML model training will be prepared after the data exploration stage. Data exploration may reveal unnecessary tokens or anomalies in the data. Any unnecessary tokens that are not informative must be removed, which will also impact the creation of n-grams. Thus, the final DTM must be made after further analyses and operations, such as exploratory data analysis and feature selection.

11 DATA EXPLORATION

- ☐ describe preparing, wrangling, and exploring text-based data for financial forecasting

Exploratory Data Analysis

Exploratory data analysis (EDA) performed on text data provides insights on word distribution in the text. Word counts from all the sentences are computed. These word counts can be used to examine outlier tokens—words that are most commonly

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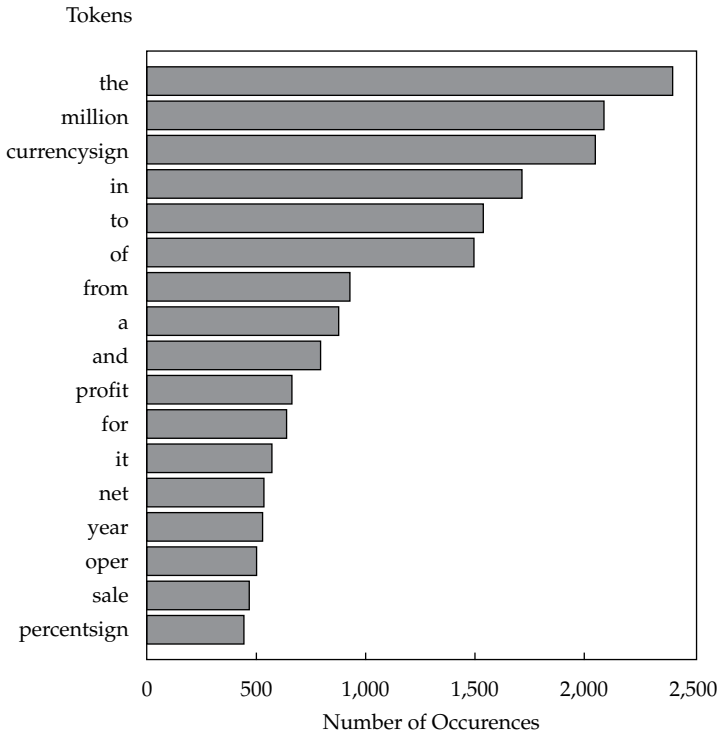
and least commonly present in the texts. The most frequent word occurrences in all sentences from the dataset are shown in Exhibit 32. These common words will be removed during the feature selection step. Notably, the tokens "million" and "currencysign" occur frequently due to the financial nature of the data.

Exhibit 32: Most Frequently Used Tokens in the Corpus



The most frequent word occurrences in the sentences in the negative sentiment and the positive sentiment classes are shown in Exhibit 33. The most commonly occurring words are similar for both sentiment classes, meaning that they are not useful in discriminating between the two sentiment classes. This finding demonstrates the utility of removing the most commonly used tokens from the BOW.

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Exhibit 33: Most Frequently Used Tokens in Two Sentiment Classes of the Corpus

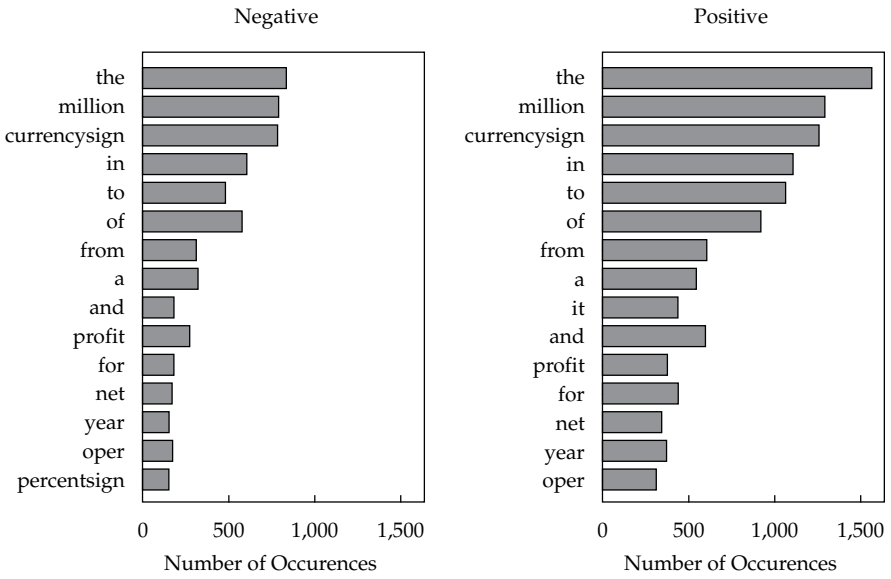
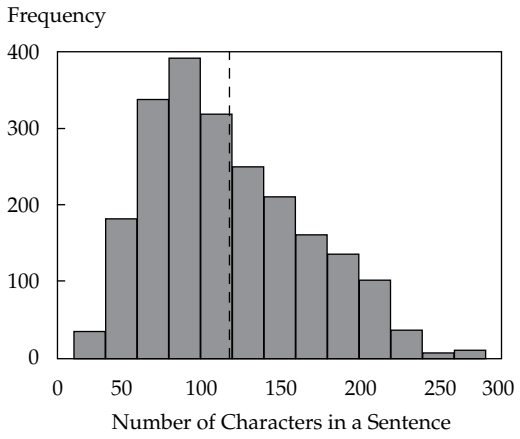


Exhibit 34 shows a histogram of sentence length distribution. **Sentence length** is defined as the number of characters, including spaces, in a sentence. The longest sentence has 273 characters; the shortest sentence has 26 characters; and the average number of characters is about 120 (indicated by the vertical line). Although this distribution does not have any direct impact on model training, this histogram visually demonstrates the range of sentence lengths and helps identify any extremely long or short sentences. This histogram does not appear unusual, so no outlier sentences need to be removed.

Exhibit 34: Histogram of Sentence Lengths with Mean Sentence Length



Word clouds are a convenient method of visualizing the text data because they enable rapid comprehension of a large number of tokens and their corresponding weights. Exhibit 35 shows a word cloud for all the sentences in the corpus. The font sizes

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of the words are proportionate to the number of occurrences of each word in the corpus. Similarly, Exhibit 36 shows the word cloud divided into two halves: one half representing negative sentiment class sentences (upper half); one half representing positive sentiment class sentences (lower half). Notably, some highly discriminative stems and words, such as "decreas" and "down" in the negative half and "increas" and "rose" in the positive half, are present. The feature selection process will eliminate common words and highlight useful words for better model training.

Exhibit 35: Word Cloud of Entire Corpus

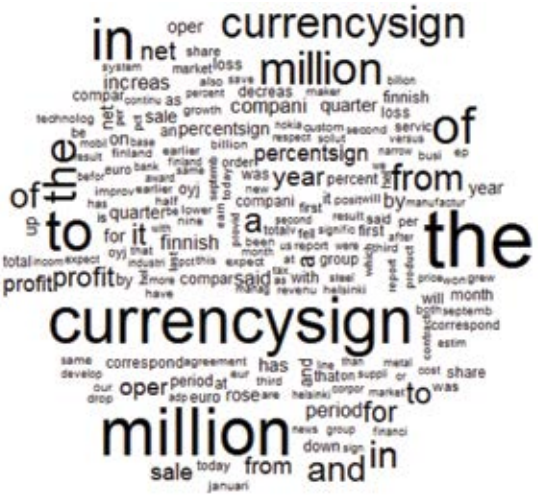

Exhibit 36: Word Cloud Divided by Two Sub-Groups of the Corpus

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Feature Selection

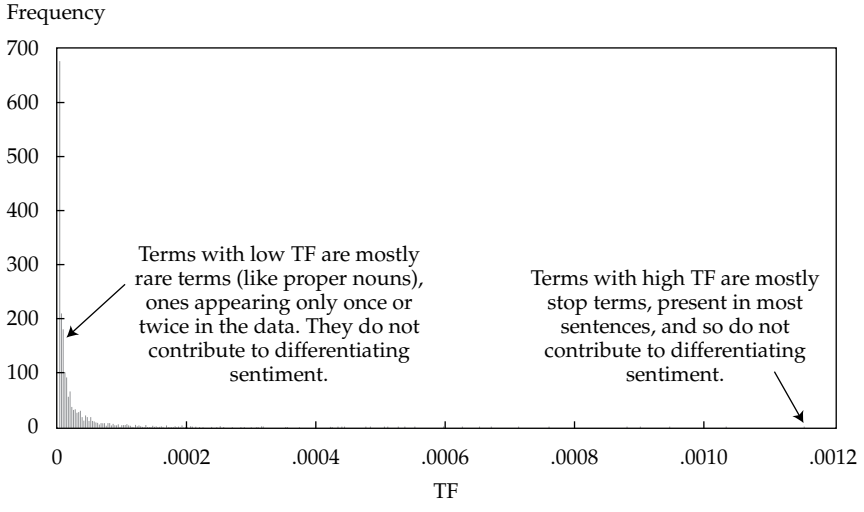
Exploratory data analysis revealed the most frequent tokens in the texts that could potentially add noise to this ML model training process. In addition to common tokens, many rarely occurring tokens, often proper nouns (i.e., names), are not informative for understanding the sentiment of the sentence. Further analyses must be conducted to decide which words to eliminate. Feature selection for text data involves keeping the useful tokens in the BOW that are informative and help to discriminate different classes of texts—those with positive sentiment and those with negative sentiment. At this point, a total of 44,151 non-unique tokens are in the 2,180 sentences.

Frequency analysis on the processed text data helps in filtering unnecessary tokens (or features) by quantifying how important tokens are in a sentence and in the corpus as a whole. Term frequency (TF) at the corpus level—also known as **collection frequency (CF)**—is the number of times a given word appears in the whole corpus (i.e., collection of sentences) divided by the total number of words in the corpus. Term frequency can be calculated and examined to identify outlier words. Exhibit 37 shows the descriptive statistics of term frequency for the words at the collection level. The statistics of TF range between 0 and 1 because TF values are ratios of total occurrences of a particular word to total number of words in the collection. A sample of words with the highest TF and lowest TF values is also shown to gain insight into what kinds of words occur at these extreme frequencies.

Exhibit 37: Summary Statistics of TF for Words at the Collection Level, Sample Words with High and Low TF Values, and Histogram of TF Values

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
2.265e-05	2.265e-05	4.530e-05	3.741e-04	1.585e-04	5.429e-02

word	TF	word	TF
<chr>	<dbl>	<chr>	<dbl>
the	0.05429096	yet	2.264954e-05
million	0.04722430	yihn	2.264954e-05
currencysign	0.04627302	young	2.264954e-05
in	0.03870807	zahariev	2.264954e-05
to	0.03476705	zone	2.264954e-05
of	0.03377047	zoo	2.264954e-05



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Calculating highest and lowest TFs at the collection level is a general strategy to identify noisy terms. The histogram in Exhibit 37 shows a long tail to the right, which represents common terms that must be removed. The high frequency bars on the left show that there are also many rare terms (e.g., ones appearing only once or twice across the data). Such rare terms do not appear enough to be used as meaningful features and are often removed. The words with the highest TF are mostly stop words that are not useful because they are present in most of the sentences and thus do not contribute to differentiating the sentiment embedded in the text. The words with the lowest TF values are mostly proper nouns or sparse terms that are also not important to the meaning of the text. In this example, after careful examination of words with extreme frequencies, the words with high TF values (>99.5th percentile, 14 words) and low TF values (<30th percentile, 714 words) are removed before forming the final document term matrix (DTM). Exhibit 38 shows the 14 words with the highest TF values (>99.5th percentile) that are the custom stop words for this project.

Exhibit 38: Fourteen Custom Stop Words for the Project

"the"	"million"	"currencysign"	"in"	"to"	"of"	"from"
"and"	"profit"	"for"	"it"	"not"	"year"	"a"

To construct a DTM for ML training, different TF measures need to be computed to fill in the cells of the DTM. Exhibit 39 displays part of a TF measures table that is computed for the text data before the removal of custom stop words.

Exhibit 39: Sample Output of Term Frequency (TF) Measures Table

SentenceNo	TotalWordsInSentence	Word	TotalWordCount	WordCountInSentence	SentenceCountWithWord	TF	DF	IDF	TFIDF
<int>	<int>	<chr>	<int>	<int>	<int>	<dbl>	<dbl>	<dbl>	<dbl>
624	34	a	873	6	687	0.1764706	0.3151376	1.1547459	0.20377868
701	39	the	2397	6	1453	0.1538462	0.6665138	0.4056945	0.06241454
1826	34	a	873	6	687	0.1764706	0.3151376	1.1547459	0.20377868
1963	39	the	2397	6	1453	0.1538462	0.6665138	0.4056945	0.06241454
128	30	of	1491	5	984	0.1666667	0.4513761	0.7954543	0.13257571
223	37	the	2397	5	1453	0.1351351	0.6665138	0.4056945	0.05482358

The columns of the term frequency measures table are as follows:

1. *SentenceNo*: A unique identification number assigned to each sentence in the order they are present in the original dataset. For example, sentence number 701 is a sentence in row 701 from the data table: "*the airlin estim that the cancel of it flight due to the closur of european airspac and the process of recommenc traffic have caus a the compani a loss of currencysign million includ the cost of strand passeng accommod*."
2. *TotalWordsInSentence*: Count of total number of words present in the sentence. For example, sentence number 701 has a total of 39 words.
3. *Word*: A word token that is present in the corresponding sentence.
4. *TotalWordCount*: Total number of occurrences of the word in the entire corpus or collection. For example, the token "the" occurs 2,397 times in the whole collection of sentences. The following equation can be used to compute TF at the collection level:

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TF (Collection Level)

=

TotalWordCount/Total number of words in collection. (10)

The TF of the word "the" at the collection level is calculated as 2,397/44,151 = 0.05429096. Note that this result was seen previously in Exhibit 37.

5. *WordCountInSentence*: Number of times the token is present in the corresponding sentence. For example, token "the" is present six times in sentence number 701.
6. *SentenceCountWithWord*: Number of sentences in which the word is present. For example, the token "the" is present in 1,453 sentences.
7. *TF (Term Frequency) at Sentence Level*: Number of times a word is present in a sentence divided by the total number of words in that sentence. The following equation can be used to compute TF at the sentence level:

TF (Sentence Level)

=

WordCountInSentence/TotalWordsInSentence. (11)

For example, TF at the sentence level for the word "the" in sentences number 701 and 223 is calculated as 6/39 = 0.1538462 and 5/37 = 0.1351351, respectively.

8. *DF (Document Frequency)*: Defined as the number of documents (i.e., sentences) that contain a given word divided by the total number of sentences (here, 2,180). Document frequency is important since words frequently occurring across sentences provide no differentiating information in each sentence. The following equation can be used to compute DF:

DF = SentenceCountWithWord/Total number of sentences. (12)

For example, DF of the word "the" is 1,453/2,180 = 0.6665138; so, 66.7% of the sentences contain the word "the." A high DF indicates high word frequency in the text.

9. *IDF (Inverse Document Frequency)*: A relative measure of how unique a term is across the entire corpus. Its meaning is not directly related to the size of the corpus. The following equation can be used to compute IDF:

IDF = log(1/DF). (13)

For example, IDF of the word "the" is log(1/0.6665138) = 0.4056945. A low IDF indicates high word frequency in the text.

10. *TF–IDF*: To get a complete representation of the value of each word, TF at the *sentence level* is multiplied by the IDF of a word across the entire dataset. Higher TF–IDF values indicate words that appear more frequently within a smaller number of documents. This signifies relatively more unique terms that are important. Conversely, a low TF–IDF value indicates terms that appear in many documents. TF–IDF values can be useful in measuring the key terms across a compilation of documents and can serve as word feature values for training an ML model. The following equation can be used to compute TF–IDF:

TF–IDF = TF × IDF. (14)

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For example, TF–IDF of the token "of" is calculated as 0.1666667 × 0.7954543 = 0.13257571.

Similarly, Exhibit 40 shows high TF–IDF words for the text data before the removal of custom stop words.

Exhibit 40: Sample Output of High TF–IDF Words

SentenceNo	TotalWordsInSentence	Word	TotalWordCount	WordCountInSentence	SentenceCountWithWord	TF	DF	IDF	TFIDF
<int>	<int>	<chr>	<int>	<int>	<int>	<dbl>	<dbl>	<dbl>	<dbl>
28	7	risen	3	1	3	0.1428571	0.0013761468	6.588468	0.9412097
830	7	diminish	2	1	2	0.1428571	0.0009174312	6.993933	0.9991333
1368	9	great	4	1	4	0.1111111	0.0018348624	6.300786	0.7000873
1848	8	injuri	1	1	1	0.1250000	0.0004587156	7.687080	0.9608850
1912	7	cheaper	1	1	1	0.1428571	0.0004587156	7.687080	1.0981543
1952	6	argument	1	1	1	0.1666667	0.0004587156	7.687080	1.2811800

TF or TF–IDF values are placed at the intersection of sentences (rows) and terms (columns) of the document term matrix. For this project, TF values are used for the DTM as the texts are sentences rather than paragraphs or other larger bodies of text. TF–IDF values vary by the *number* of documents in the dataset; therefore, the model performance can vary when applied to a dataset with just a few documents. In addition to removing custom stop words and sparse terms, single character letters are also eliminated because they do not add any value to the sentiment significance.

Feature Engineering

N-grams are used as a feature engineering process in this project. Use of n-grams helps to understand the sentiment of a sentence as a whole. As mentioned previously, the objective of this project is to predict sentiment class (positive and negative) from financial texts. Both unigram and bigrams are implemented, and the BOW is created from them. Bigram tokens are helpful for keeping negations intact in the text, which is vital for sentiment prediction. For example, the tokens "not" and "good" or "no" and "longer" can be formed into single tokens, now bigrams, such as "not_good" and "no_longer." These and similar tokens can be useful during ML model training and can improve model performance. Exhibit 41 shows a sample of 100 words from the BOW containing both unigram and bigram tokens after removal of custom stop words, sparse terms, and single characters. Note that the BOW contains such tokens as increas, loss, loss_prior, oper_rose, tax_loss, and sale_increas. Such tokens are informative about the embedded sentiment in the texts and are useful for training an ML model. The corresponding word frequency measures for the document term matrix are computed based on this new BOW.

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Exhibit 41: One-Hundred Sample Tokens from Final BOW of Entire Corpus

"last"	"last_quarter"	"quarter"	"quarter_componenta"	"componenta"
"componenta_sale"	"sale"	"sale_doubl"	"doubl"	"doubl_same"
"same"	"same_period"	"period"	"period_earlier"	"earlier"
"earlier_while"	"while"	"while_move"	"move"	"move_zero"
"zero"	"zero_pre"	"pre"	"pre_tax"	"tax"
"tax_pre"	"tax_loss"	"loss"	"third"	"third_quarter"
"quarter_sale"	"sale_increas"	"increas"	"increas_by"	"by"
"by_percentsign"	"percentsign"	"percentsign_oper"	"oper"	"oper_by"
"oper_rose"	"rose"	"rose_correspond"	"correspond"	"correspond_period"
"period_repres"	"repres"	"repres_percentsign"	"percentsign_sale"	"oper_total"
"total"	"total_up"	"up"	"up_repres"	"finnish"
"finnish_talentum"	"talentum"	"talentum_report"	"report"	"report_oper"
"oper_increas"	"increas_sale"	"sale_total"	"cloth"	"cloth_retail"
"retail"	"retail_chain"	"chain"	"chain_sepp"	"sepp"
"sepp_ls"	"ls"	"ls_sale"	"consolid"	"consolid_sale"
"incres_percentsign"	"percentsign_reach"	"reach"	"reach_while"	"while_oper"
"oper_amount"	"amount"	"amount_compar"	"compar"	"compar_loss"
"loss_prior"	"prior"	"prior_period"	"foundri"	"foundri_divis"
"divis"	"divis_report"	"report_sale"	"percentsign_correspond"	"period_sale"
"sale_machin"	"machin"	"machin_shop"	"shop"	"shop_divis"

EXAMPLE 6

Calculating and Interpreting Term Frequency Measures

Data scientists Jack and Jill are using financial text data to develop sentiment indicators for forecasting future stock price movements. They have assembled a BOW from the corpus of text being examined and have pulled the following abbreviated term frequency measures tables.

Exhibit 42: Term Frequency Measures Table 1

SentenceNo	TotalWordsInSentence	Word	TotalWordCount	WordCountInSentence	SentenceCountWithWord
<int>	<int>	<chr>	<int>	<int>	<int>
624	34	a	873	6	687
701	39	the	2397	6	1453
1826	34	a	873	6	687
1963	39	the	2397	6	1453
128	30	of	1491	5	984
223	37	the	2397	5	1453

Exhibit 43: Term Frequency Measures Table 2

SentenceNo	TotalWordsInSentence	Word	TotalWordCount	WordCountInSentence	SentenceCountWithWord
<int>	<int>	<chr>	<int>	<int>	<int>
28	7	risen	3	1	3
830	7	diminish	2	1	2
1368	9	great	4	1	4
1848	8	injuri	1	1	1
1912	7	cheaper	1	1	1
1952	6	argument	1	1	1

1. Determine and interpret term frequency (TF) at the collection level and at the sentence level for the word (i.e., token) "a" in sentence 1,826 in term frequency measures Table 1 and then for the token "great" in sentence 1,368 in term frequency measures Table 2.

Solution:

TF at the collection level is calculated using Equation 10:

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TF (Collection Level) = TotalWordCount/Total number of words in collection.

For token "a" in sentence 1,826 (Table 1), TF (Collection Level) is 873/44,151 = 0.019773 or 1.977%.For token "great" in sentence 1,368 (Table 2), TF (Collection Level) is 4/44,151 = 0.000091 or 0.009%.TF at the collection level is an indicator of the frequency, in percentage terms, that a token is used throughout the whole collection of texts (here, 44,151). It is useful for identifying outlier words: Tokens with highest TF values are mostly stop words that do not contribute to differentiating the sentiment embedded in the text (such as "a"), and tokens with lowest TF values are mostly proper nouns or sparse terms that are also not important to the meaning of the text. Conversely, tokens with intermediate TF values potentially carry important information useful for differentiating the sentiment embedded in the text.TF at the sentence level is calculated using Equation 11:

TF (Sentence Level) = WordCountInSentence/TotalWordsInSentence.

For token "a" in sentence 1,826, TF (Sentence Level) is 6/34 = 0.176471 or 17.647%.
For token "great" in sentence 1,368, TF (Sentence Level) is 1/9 = 0.111111 or 11.111%.
TF at the sentence level is an indicator of the frequency, in percentage terms, that a token is used in a particular sentence (i.e., instance). Therefore, it is useful for understanding the importance of the specific token in a given sentence.

2. Determine and interpret TF–IDF (term frequency–inverse document frequency) for the word "a" in sentence 1,826 in term frequency measures Table 1 and then for the token "great" in sentence 1,368 in term frequency measures Table 2.

Solution:

To calculate TF–IDF, besides TF at the sentence level, document frequency (DF) and inverse document frequency (IDF) are also required.

DF is the number of documents (i.e., sentences) that contain a given word divided by the total number of sentences in the corpus (here, 2,180). DF is calculated using Equation 12:

DF = SentenceCountWithWord/Total number of sentences.

For token "a" in sentence 1,826, DF is 687/2,180 = 0.315138 or 31.514%.
For token "great" in sentence 1,368, DF is 4/2,180 = 0.001835 or 0.184%.
Document frequency is important since tokens occurring frequently across sentences (such as "a") provide no differentiating information in each sentence. Tokens occurring less frequently across sentences (such as "great"), however, may provide useful differentiating information.
IDF is a relative measure of how important a term is across the entire corpus (i.e., collection of texts/sentences). IDF is calculated using Equation 13:

IDF = log(1/DF).

For token "a" in sentence 1,826, IDF is log(1/0.315138) = 1.154746.
For token "great" in sentence 1,368, IDF is log(1/0.001835) = 6.300786.
Using TF and IDF, TF–IDF can now be calculated using Equation 14:

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TF–IDF = TF × IDF.

For token "a" in sentence 1,826, TF–IDF = 0.176471 × 1.154746 = 0.203779, or 20.378%.
For token "great" in sentence 1,368, TF–IDF = 0.111111 × 6.300786 = 0.700087, or 70.009%.
As TF–IDF combines TF at the *sentence level* with IDF across the entire corpus, it provides a complete representation of the value of each word. A high TF–IDF value indicates the word appears many times within a small number of documents, signifying an important yet unique term within a sentence (such as "great"). A low TF–IDF value indicates tokens that appear in most of the sentences and are not discriminative (such as "a"). TF–IDF values are useful in extracting the key terms in a document for use as features for training an ML model.

12 MODEL TRAINING

- describe preparing, wrangling, and exploring text-based data for financial forecasting

The sentiment class labels (positive and negative) constitute the target variable (y) for model training. They are relabeled as 1 (for positive) and 0 (for negative) to enable calculating the performance metrics, such as receiver operating characteristic (ROC) curve and area under the curve (AUC) from the trained model results. The master dataset that has been cleansed and preprocessed is partitioned into three separate sets: 1) training set; 2) cross-validation (CV) set; and 3) test set. These are in the ratio of 60:20:20, respectively (following common practice). For splitting, simple random sampling is applied within levels of the target variable to balance the class distributions within the splits. The final DTM is built using the sentences (rows), which are the instances, and resulting tokens (columns), which are the feature variables, from the BOW of the training dataset. The final BOW consists of unigram and bigram tokens from the sentences in the training corpus only. The DTM is then filled in with resultant TF values of the tokens from the training corpus.

Similarly, the DTMs for the CV set and the test set are built using tokens from the final training BOW for tuning, validating, and testing of the model. To be clear, the final BOW from the training corpus is used for building DTMs across all the splits because the model has been trained on that final BOW. Thus, the columns (think, features) of all three DTMs are the same, but the number of rows varies because a different number of sentences are in each split. The DTMs are filled with resultant term frequency values calculated using sentences in the corpuses of the respective splits—sentences from the CV set corpus and sentences from the test set corpus. Exhibit 44 tabulates the summary of dimensions of the data splits and their uses in the model training process. As mentioned, the columns of DTMs for the splits are the same, equal to the number of unique tokens (i.e., features) from the final training corpus BOW, which is 9,188. Note that this number of unique tokens (9,188) differs from that in the master corpus (11,501) based on the sentences that are included in the training corpus after the random sampling.

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Exhibit 44: Summary of the Three Data Splits

Corpus	Split %	Number of Sentences	DTM Dimensions	Purpose
Master	100%	2180	2180 × 11501	Used for data exploration
Training	60%	1309	1309 × 9188	Used for ML model training
CV	20%	435	435 × 9188	Used for tuning and validating the trained model
Test	20%	436	436 × 9188	Used for testing the trained, tuned, and validated model

Method Selection

Alternative ML methods, including SVM, decision trees, and logistic regression, were examined because these techniques are all considered potentially suitable for this particular task (i.e., supervised learning), type of data (i.e., text), and size of data (i.e., wider data with many potential variables). The SVM and logistic regression methods appeared to offer better performance than decision trees. For brevity, we discuss logistic regression in the remainder of the chapter. Logistic regression was used to train the model, using the training corpus DTM containing 1,309 sentences. As a reminder, in this project texts are the sentences and the classifications are positive and negative sentiment classes (labeled 1 and 0, respectively). The tokens are feature variables, and the sentiment class is the target variable. Text data typically contain thousands of tokens. These result in sparse DTMs because each column represents a token feature and the values are mostly zeros (i.e., not all the tokens are present in every text). Logistic regression can deal with such sparse training data because the regression coefficients will be close to zero for tokens that are not present in a significant number of sentences. This allows the model to ignore a large number of minimally useful features. Regularization further helps lower the coefficients when the features rarely occur and do not contribute to the model training.

Logistic regression is applied on the final training DTM for model training. As this method uses maximum likelihood estimation, the output of the logistic model is a probability value ranging from 0 to 1. However, because the target variable is binary, coefficients from the logistic regression model are not directly used to predict the value of the target variable. Rather, a mathematical function uses the logistic regression coefficient (β) to calculate probability (p) of sentences having positive sentiment (y = 1).[3] If p for a sentence is 0.90, there is a 90% likelihood that the sentence has positive sentiment. Theoretically, the sentences with p > 0.50 likely have positive sentiment. Because this is not always true in practice, however, it is important to find an ideal threshold value of p. We elaborate on this point in a subsequent example. The threshold value is a cutoff point for p values, and the ideal threshold p value is influenced by the dataset and model training. When the p values (i.e., probability of sentences having positive sentiment) of sentences are above this ideal threshold p value, then the sentences are *highly* likely to have positive sentiment (y = 1). The ideal threshold p value is estimated heuristically using performance metrics and ROC curves, as will be demonstrated shortly.

3 This mathematical function is an exponential function of the form: $P(y = 1) = \frac{1}{1 + \exp^{-(\beta_0+\beta_1 x_1+\beta_2 x_2+\cdots+\beta_n x_n)}}$

where the βs are the logistic regression coefficients.

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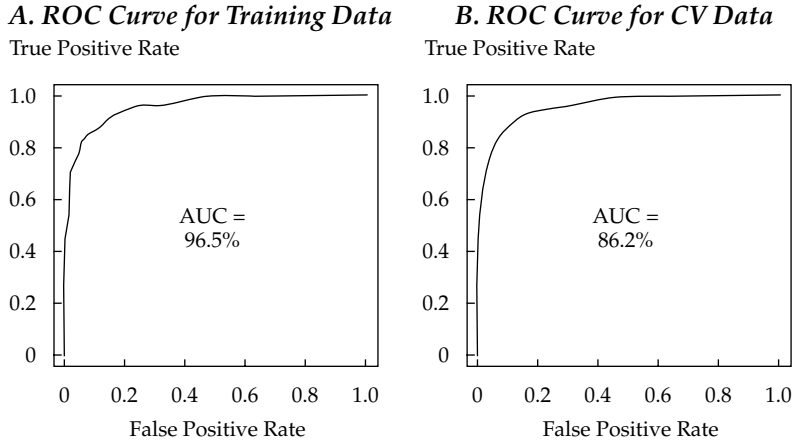
Performance Evaluation and Tuning

The trained ML model is used to predict the sentiments of the sentences in the training and CV DTMs. Exhibit 45 displays the ROC curves for the training (Panel A) and CV (Panel B) data. Remember that the x-axis is false positive rate, FP/(TN + FP), and the y-axis is true positive rate, TP/(TP + FN). As the model is trained using the training DTM, it clearly performs well on the same training data (so there is no concern about underfitting) but does not perform as well on the CV data. This is apparent as the ROC curves are significantly different between the training and CV datasets. The AUC is 96.5% on training data and 86.2% on CV data. This finding suggests that the model performs comparatively poorly (with a higher rate of error or misclassification) on the CV data when compared to training data. Thus, the implication is that the model is overfitted.

Exhibit 45: ROC Curves of Model Results for Training and CV Data Before Regularization



As the model is overfitted, least absolute shrinkage and selection operator (LASSO) regularization is applied to the logistic regression. LASSO regularization penalizes the coefficients of the logistic regression to prevent overfitting of the model. The penalized regression will select the tokens (features) that have statistically significant (i.e., non-zero) coefficients and that contribute to the model fit; LASSO does this while disregarding the other tokens. Exhibit 46 shows the ROC curves for the new model that uses regularized logistic regression. The ROC curves look similar for model performance on both datasets, with an AUC of 95.7% on the training dataset (Panel A) and 94.8% on the CV dataset (Panel B). These findings suggest that the model performs similarly on both training and CV data and thus indicate a good fitting model (one that is not overfitted).

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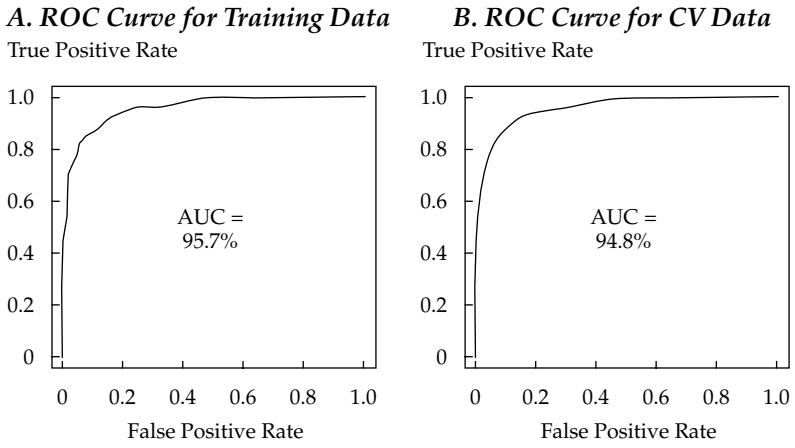
Exhibit 46: ROC Curves of Model Results for Training and CV Data After Regularization



Regularization along with careful feature selection help to prevent overfitting in logistic regression models. Another model was trained using all token features, including stop words, sparse terms, and single characters, with no regularization. That model showed an AUC of 99.1% when applied on the training dataset and an AUC of 89.4% when applied on the CV dataset, suggesting that the model is overfitting. As the AUC values in all of the models discussed are not far from 100%, these models are clearly not underfitting. In sum, the final ML model for this project uses logistic regression with LASSO regularization.

To further evaluate the model, error analysis is conducted by calculating a confusion matrix using the ML model results from the cross-validation dataset. The threshold p value of 0.5 is used as a cutoff point. When target value p > 0.5, the prediction is assumed to be $y = 1$ (meaning, positive sentiment). Otherwise, the prediction is assumed to be $y = 0$ (negative sentiment). A confusion matrix, with performance metrics and overall scores for the model results using the CV data, is shown in Exhibit 47.

Exhibit 47: Confusion Matrix of Model Results for CV Data with Threshold p Value = 0.50

Confusion Matrix for CV Data with Threshold = 0.5

		Actual Training Labels	
		Class "1"	Class "0"
Predicted Results	Class "1"	284 (TP)	38 (FP)
	Class "0"	7 (FN)	106 (TN)

Performance Metrics

TP = 284, FP = 38, FN = 7, TN = 106
P = 284 / (284+38) = 0.88
R = 284 / (284+7) = 0.98
F1 Score = (2 × 0.88 × 0.98) / (0.88 + 0.98) = 0.93
Accuracy = (284 + 106) / (284 + 38 + 106 + 7) = 0.90

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The model accuracy is 90% with a theoretically suggested (default) threshold p value of 0.5. The CV data are used to tune the threshold value for best model performance. Various p values from 0.01 to 0.99 are systematically evaluated individually, and confusion matrixes and performance metrics are calculated using each of these p values. Based on these metrics, the p value resulting in the highest model accuracy is selected as the ideal threshold p value. However, there are often trade-offs: Minimizing false positives (FPs) comes at a cost of increasing false negatives (FNs), and vice versa. Prioritizing various performance statistics (e.g., precision versus recall) depends on the context and relative consequences of FP and FN on the project applications. In this project, the values of negative sentiment and positive sentiment sentences are assumed to be equal, thus the impacts of FP and FN are also equal. It is common practice to simulate many model results using different threshold p values and to search for maximized accuracy and F1 statistics that minimize these trade-offs. As noted earlier, accuracy and F1 scores are overall performance measures that give equal weight to FP and FN.

Exhibit 48 shows the overall performance measures (i.e., F1 score and accuracy) for various threshold p values. The threshold p value that results in the highest accuracy and F1 score can now be identified. From the charts in Exhibit 47, the ideal threshold p value appears to be around 0.60. To investigate further, a table of performance measures (i.e., precision, recall, F1 score, and accuracy) is generated for a series of threshold p values ranging from 0.45 to 0.75. The table in Exhibit 49 demonstrates that threshold p values between 0.60 and 0.63 result in the highest accuracy and F1 score for the CV dataset. As a result of this analysis, a final threshold p value of 0.60 is selected.

Exhibit 48: Threshold Values Versus Overall Performance Measures

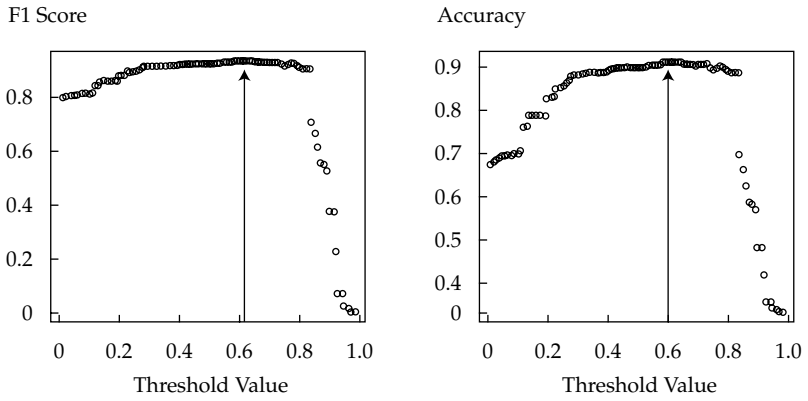

Exhibit 49: Performance Measures of the Model for a Series of Threshold Values

Threshold	Precision	Recall	F1	Accuracy
0.45	0.8750000	0.986254296	0.927302100	0.8965517
0.46	0.8827160	0.982817869	0.930081301	0.9011494
0.47	0.8827160	0.982817869	0.930081301	0.9011494
0.48	0.8819876	0.975945017	0.926590538	0.8965517
0.49	0.8819876	0.975945017	0.926590538	0.8965517

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Threshold	Precision	Recall	F1	Accuracy
0.50	0.8819876	0.975945017	0.926590538	0.8965517
0.51	0.8819876	0.975945017	0.926590538	0.8965517
0.52	0.8819876	0.975945017	0.926590538	0.8965517
0.53	0.8902821	0.975945017	0.931147541	0.9034483
0.54	0.8930818	0.975945017	0.932676519	0.9057471
0.55	0.8930818	0.975945017	0.932676519	0.9057471
0.56	0.8958991	0.975945017	0.934210526	0.9080460
0.57	0.8958991	0.975945017	0.934210526	0.9080460
0.58	0.8958991	9.975945017	0.934210526	0.9080460
0.59	0.9015873	0.975945017	0.937293729	0.9126437
0.60	0.9044586	0.975945017	0.938842975	0.9149425
0.61	0.9044586	0.975945017	0.938842975	0.9149425
0.62	0.9044586	0.975945017	0.938842975	0.9149425
0.63	0.9041534	0.972508591	0.937086093	0.9126437
0.64	0.9041534	0.972508591	0.937086093	0.9126537
0.65	0.9041534	0.972508591	0.937086093	0.9126437
0.66	0.9035370	0.965635739	0.933554817	0.9080460
0.67	0.9035370	0.965635739	0.933554817	0.9080460
0.68	0.9064516	0.965635739	0.935108153	0.9103448
0.69	0.9064516	0.965635739	0.935108153	0.9103448
0.70	0.9061489	0.962199313	0.933333333	0.9080460
0.71	0.9061489	0.962199313	0.933333333	0.9080460
0.72	0.9090909	0.962199313	0.934891486	0.9103448
0.73	0.9090909	0.962199313	0.934891486	0.9103448
0.74	0.9078947	0.948453608	0.927731092	0.9011494
0.75	0.9072848	0.941580756	0.924114671	0.8965517

** The shaded row shows the selected threshold p value (0.60) and the performance metrics for the selected model.*

Finally, the confusion matrix using the ideal threshold p value of 0.60 is constructed to observe the performance of the final model. When target value $p > 0.60$, the prediction is assumed to be $y = 1$ (indicating positive sentiment); otherwise, the prediction is assumed to be $y = 0$ (negative sentiment). The confusion matrix for the CV data is shown in Exhibit 50. It is clear that the model performance metrics have improved in the final model compared to the earliest case when the threshold p value was 0.50. Now, accuracy and F1 score have both increased by one percentage point to 91% and 94%, respectively, while precision has increased by two percentage points to 90%.

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Exhibit 50: Confusion Matrix of Model Results for CV Data with Threshold p Value = 0.60

Confusion Matrix for CV Data with Threshold = 0.6

		Actual Training Labels	
		Class "1"	Class "0"
Predicted Results	Class "1"	284 (TP)	30 (FP)
	Class "0"	7 (FN)	114 (TN)

Performance Metrics

TP = 284, FP = 30, FN = 7, TN = 114
P = 284 / (284+30) = 0.90
R = 284 / (284+7) = 0.98
F1 Score = (2 × 0.90 × 0.98) / (0.90 + 0.98) = 0.94
Accuracy = (284 + 114) / (284 + 30 + 114 + 7) = 0.91

13 RESULTS AND INTERPRETATION

- ☐ describe preparing, wrangling, and exploring text-based data for financial forecasting

The final ML model with the appropriate threshold p value has been validated and is now ready for use. The model can be used to predict the sentiment of new sentences from the test data corpus as well as new sentences from similar financial text data sources, such as news wires, earnings call transcripts, and quarterly financial reports. The final model is a collection of penalized regression coefficients for unigram and bigram tokens from the BOW of the training corpus. To use the model to predict the sentiment of new sentences, tokenization and identical cleansing and preprocessing operations must be performed on the new sentences. All the processes performed on the training data must be performed on the new data to which the model will be applied (as was done for the test dataset). The model will use the trained penalized regression coefficients on the term frequency (TF) values of the tokens in the document term matrix (DTM) of the new sentences and will determine the target value (p). The columns of the DTM of the new sentences are the same as those of the training DTM, but the TF values are calculated based on the test corpus. Using the threshold p value of 0.60, the sentiment class for each sentence in the test corpus will be predicted.

The model is now applied on the test data that contains 436 sentences. Note that the test data were not used to train or validate/tune the model and are new to the model. The test data were preprocessed identically to the training and CV data while a part of the master corpus. The model is then applied to the test DTM, and the results are obtained. Exhibit 51 displays 30 sample results from the test corpus. The results table contains cleansed and preprocessed sentences, actual sentiment, target p values from the model, and predicted sentiment. Note that this sample contains three cases of misclassification: the 10th sentence (text), where p = 0.46; the 26th text, where p = 0.77; and the 30th text, where p = 0.71. Therefore, accuracy of this 30-text sample is 27/30 = 90%.

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Exhibit 51: Thirty Sample Results of Test Data

Sentence	Sentiment	p	Predicted Sentiment
exclude non recur item pre tax surg percentsign	1	0.81	1
adp news feb finnish retail kesko oyj hel kesbv said today total sale exclud valu ad tax vat stood at januari down percentsign on yea	0	0.12	0
india trade with russia current stand at four billion dollar grow per cent fiscal	1	0.83	1
refin margin was bbl combar bbl prior	1	0.81	1
scania morgan Stanley lift share target on swedish heavi duti truck bus maker scania ab crown euro crown euro	1	0.83	1
deal is like bring save	1	0.83	1
will also strengthen ruukki offshore busi	1	0.83	1
last week finnish metl technolog group announc plan sell more than percent technolog unit further compani strategy goal becom world largest stainless steel maker	1	0.83	1
nest oil board propos dividend full compar with ago	1	0.81	1
pre tax loss total compar loss first quarter	1	0.46	0
pretax total compar loss fourth quarter	1	0.74	1
re use back into pet bottle has also steadili increas rate use strap tape has pick up again after dip pector said previous	1	0.95	1
satama sale would be higher than befor	1	0.83	1
octob finnish wood product technolog supplier raut oyj hel rutav said today swung first nine month versus loss same period earlier	1	0.79	1
ebit total compar loss correspond period	1	0.74	1
finnish consum packag manufactur huhtamaki oyj said swung euro first nine month loss euro same period	1	0.77	1
finnish dental care group oral hammaslaakarit oyj post total euro first nine month versus loss euro same period	1	0.79	1
finnish silicon water manufactur okmet oyj said swung euro first nine month loss euro earlier	1	0.77	1
adp news feb finnish print circuit board pcb maker aspocomp group oyj hel acg said today swung versus loss	1	0.79	1
mn pretax third quarter	1	0.83	1
oper total compar correspond period	1	0.81	1
raut post euro third quarter compar loss euro correspond period	1	0.74	1
russian export duti will active harvest finland sale russia will increas also	1	0.91	1
compani expect sale signific increas	1	0.91	1
compani amount ee which was percentsign more than	1	0.81	1
third quarter fiscal efor swung loss versus correspond period fiscal	0	0.77	1
acando ab acanb ss fell percent kronor lowest close sinc dec	0	0.20	0
compani oper loss total compar	0	0.27	0
last paseng flew airlin down percent	0	0.12	0
loss after financi item total compar correspond period	0	0.71	1

Exhibit 52 shows the confusion matrix for the test data. Accuracy and F1 score are 90% and 93%, respectively, while precision and recall are 89% and 98%, respectively. Therefore, it is apparent that the model performs similarly on the training, CV, and

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test datasets. These findings suggest that the model is robust and is not overfitting. They also suggest that the model should generalize well out-of-sample and can thus be used to predict the sentiment classes for new sentences from similar financial text data sources. Of course, these new text data must first be subjected to identical tokenization, cleansing, and preprocessing as done for the training dataset.

Exhibit 52: Confusion Matrix of Model Results for Test Data with Threshold p Value = 0.60

Confusion Matrix for Test Data

		Actual Training Labels	
		Class "1"	Class "0"
Predicted Results	Class "1"	284 (TP)	35 (FP)
	Class "0"	7 (FN)	110 (TN)

Performance Metrics

TP = 284, FP = 35, FN = 7, TN = 110
P = 284 / (284+35) = 0.89
R = 284 / (284+7) = 0.98
F1 Score = (2 × 0.89 × 0.98) / (0.89 + 0.98) = 0.93
Accuracy = (284 + 110) / (284 + 35 + 110 + 7) = 0.90

To recap, this project involves converting unstructured data (i.e., text data from financial data sources) into structured data (i.e., tokens, sentences, and term frequency values) in a document term matrix that is used as input for training, validating, and testing machine learning-based models (here, logistic regression) for predicting classification (here, sentiment classes). Similar models can be built and used in different contexts to understand the sentiment embedded in larger texts. The derived sentiment classification can be useful as a visualization tool to provide insight about the text without reading large documents. These sentiment classifications can also be used as structured input data for larger ML models that have a specific purpose, such as to predict future stock price movements.

EXAMPLE 7

Comparing Performance Metrics for Confusion Matrixes with Different Threshold p Values

In the previous analysis using the cross-validation dataset, performance measures for the sentiment classification ML model were calculated for a wide range (from 0.45 to 0.75) of threshold p values. The threshold value of 0.60 was determined to be the p value that maximizes model accuracy and F1 score; the confusion matrix for this model is shown in Exhibit 48. Use the following confusion matrixes with threshold p values of 0.75 and 0.45, A and B, respectively, to answer the following questions.

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Confusion Matrix A

Confusion Matrix for CV, Threshold = 0.75

N = 436		Actual Training Labels	
		Class "1"	Class "0"
Predicted Results	Class "1"	281	28
	Class "0"	17	110

Performance Metrics

TP = 281, FP = 28, FN = 17, TN = 110
Precision = TP/(TP + FP) = 0.91
Recall = TP/(TP + FN) = 0.94
F1 Score = HMean: Prec. & Recall = 0.93
Accuracy = (TP + TN)/N = 0.90

Confusion Matrix B

Confusion Matrix for CV, Threshold = 0.45

N = 436		Actual Training Labels	
		Class "1"	Class "0"
Predicted Results	Class "1"	281	41
	Class "0"	4	110

Performance Metrics

TP = 281, FP = 41, FN = 4, TN = 110
Precision = TP/(TP + FP) = 0.87
Recall = TP/(TP + FN) = 0.99
F1 Score = HMean: Prec. & Recall = 0.93
Accuracy = (TP + TN)/N = 0.90

1. Compare the performance metrics of confusion matrix A (using a threshold p value of 0.75) with the confusion matrix in Exhibit 48 (using a threshold p value of 0.60).

Solution:

Since confusion matrix A has fewer true positives (TPs) and fewer true negatives (TNs) than the confusion matrix in Exhibit 48 (281 vs. 284 and 110 vs. 114, respectively), confusion matrix A has lower accuracy and a lower F1 score compared to the one in Exhibit 48 (0.90 vs. 0.91 and 0.93 vs. 0.94, respectively). Also, although confusion matrix A has slightly better precision, 0.91 vs. 0.90, due to a few less false positives (FPs), it has significantly lower recall, 0.94 vs. 0.98, due to having many more false negatives (FNs), 17 vs. 7, than the confusion matrix in Exhibit 48. On balance, the ML model using the threshold p value of 0.60 is the superior model for this sentiment classification problem.

2. Compare the performance metrics of confusion matrix B (using a threshold p value of 0.45) with the confusion matrix in Exhibit 48 (using a threshold p value of 0.60).

Solution:

Confusion matrix B has the same number of TPs (281) and TNs (110) as confusion matrix A. Therefore, confusion matrix B also has lower accuracy (0.90) and a lower F1 score (0.93) compared to the one in Exhibit 48. Although confusion matrix B has slightly better recall, 0.99 vs. 0.98, due to fewer FNs, it has somewhat lower precision, 0.87 vs. 0.90, due to having many more FPs, 41 vs. 30, than the confusion matrix in Exhibit 48. Again, it is apparent that the ML model using the threshold p value of 0.60 is the better model in this sentiment classification context.

3. Contrast the performance metrics of confusion matrixes A and B, and explain the trade-offs implied between them.

Solution:

The main differences in performance metrics between confusion matrixes A and B are in precision and recall. Confusion matrix A has higher precision, at 0.91 vs. 0.87, but confusion matrix B has higher recall, at 0.99 vs. 0.94. These differences highlight the trade-off between FP (Type I error) and FN (Type II error). Precision is useful when the cost of FP is high, such as when an expensive product that is fine mistakenly fails quality inspection and is scrapped; in this case, FP should be minimized. Recall is useful when the cost of FN is high, such as when an expensive product is defective but mistakenly passes quality inspection and is sent to the customer; in this

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case, FN should be minimized. In the context of sentiment classification, FP might result in buying a stock for which sentiment is incorrectly classified as positive when it is actually negative. Conversely, FN might result in avoiding (or even shorting) a stock for which the sentiment is incorrectly classified as negative when it is actually positive. The model behind the confusion matrix in Exhibit 48 strikes a balance in the trade-off between precision and recall.

SUMMARY

In this reading, we have discussed the major steps in big data projects involving the development of machine learning (ML) models—namely, those combining textual big data with structured inputs.

- Big data—defined as data with volume, velocity, variety, and potentially lower veracity—has tremendous potential for various fintech applications, including several related to investment management.
- The main steps for traditional ML model building are conceptualization of the problem, data collection, data preparation and wrangling, data exploration, and model training.
- For textual ML model building, the first four steps differ somewhat from those used in the traditional model: Text problem formulation, text curation, text preparation and wrangling, and text exploration are typically necessary.
- For structured data, data preparation and wrangling entail data cleansing and data preprocessing. Data cleansing typically involves resolving incompleteness errors, invalidity errors, inaccuracy errors, inconsistency errors, non-uniformity errors, and duplication errors.
- Preprocessing for structured data typically involves performing the following transformations: extraction, aggregation, filtration, selection, and conversion.
- Preparation and wrangling text (unstructured) data involves a set of text-specific cleansing and preprocessing tasks. Text cleansing typically involves removing the following: html tags, punctuations, most numbers, and white spaces.
- Text preprocessing requires performing normalization that involves the following: lowercasing, removing stop words, stemming, lemmatization, creating bag-of-words (BOW) and n-grams, and organizing the BOW and n-grams into a document term matrix (DTM).
- Data exploration encompasses exploratory data analysis, feature selection, and feature engineering. Whereas histograms, box plots, and scatterplots are common techniques for exploring structured data, word clouds are an effective way to gain a high-level picture of the composition of textual content. These visualization tools help share knowledge among the team (business subject matter experts, quants, technologists, etc.) to help derive optimal solutions.
- Feature selection methods used for text data include term frequency, document frequency, chi-square test, and a mutual information measure. Feature engineering for text data includes converting numbers into tokens, creating n-grams, and using name entity recognition and parts of speech to engineer new feature variables.

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- The model training steps (method selection, performance evaluation, and model tuning) often do not differ much for structured versus unstructured data projects.
- Model selection is governed by the following factors: whether the data project involves labeled data (supervised learning) or unlabeled data (unsupervised learning); the type of data (numerical, continuous, or categorical; text data; image data; speech data; etc.); and the size of the dataset.
- Model performance evaluation involves error analysis using confusion matrixes, determining receiver operating characteristics, and calculating root mean square error.
- To carry out an error analysis for each model, a confusion matrix is created; true positives (TPs), true negatives (TNs), false positives (FPs), and false negatives (FNs) are determined. Then, the following performance metrics are calculated: accuracy, F1 score, precision, and recall. The higher the accuracy and F1 score, the better the model performance.
- To carry out receiver operating characteristic (ROC) analysis, ROC curves and area under the curve (AUC) of various models are calculated and compared. The more convex the ROC curve and the higher the AUC, the better the model performance.
- Model tuning involves managing the trade-off between model bias error, associated with underfitting, and model variance error, associated with overfitting. A fitting curve of in-sample (training sample) error and out-of-sample (cross-validation sample) error on the y-axis versus model complexity on the x-axis is useful for managing the bias vs. variance error trade-off.
- In a real-world big data project involving text data analysis for classifying and predicting sentiment of financial text for particular stocks, the text data are transformed into structured data for populating the DTM, which is then used as the input for the ML algorithm.
- To derive term frequency (TF) at the sentence level and TF–IDF, both of which can be inputs to the DTM, the following frequency measures should be used to create a term frequency measures table: TotalWordsInSentence; TotalWordCount; TermFrequency (Collection Level); WordCountInSentence; SentenceCountWithWord; Document Frequency; and Inverse Document Frequency.

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PRACTICE PROBLEMS

The following information relates to questions 1-15

Aaliyah Schultz is a fixed-income portfolio manager at Aries Investments. Schultz supervises Ameris Steele, a junior analyst.

A few years ago, Schultz developed a proprietary machine learning (ML) model that aims to predict downgrades of publicly-traded firms by bond rating agencies. The model currently relies only on structured financial data collected from different sources. Schultz thinks the model's predictive power may be improved by incorporating sentiment data derived from textual analysis of news articles and Twitter content relating to the subject companies.

Schultz and Steele meet to discuss plans for incorporating the sentiment data into the model. They discuss the differences in the steps between building ML models that use traditional structured data and building ML models that use textual big data. Steele tells Schultz:

Statement 1 The second step in building text-based ML models is text preparation and wrangling, whereas the second step in building ML models using structured data is data collection.

Statement 2 The fourth step in building both types of models encompasses data/text exploration.

Steele expresses concern about using Twitter content in the model, noting that research suggests that as much as 10%–15% of social media content is from fake accounts. Schultz tells Steele that she understands her concern but thinks the potential for model improvement outweighs the concern.

Steele begins building a model that combines the structured financial data and the sentiment data. She starts with cleansing and wrangling the raw structured financial data. Exhibit 1 presents a small sample of the raw dataset before cleansing: Each row represents data for a particular firm.

Exhibit 1: Sample of Raw Structured Data Before Cleansing

ID	Ticker	IPO Date	Industry (NAICS)	EBIT	Interest Expense	Total Debt
1	ABC	4/6/17	44	9.4	0.6	10.1
2	BCD	November 15, 2004	52	5.5	0.4	6.2
3	HIJ	26-Jun-74	54	8.9	1.2	15.8
4	KLM	14-Mar-15	72	5.7	1.5	0.0

After cleansing the data, Steele then preprocesses the dataset. She creates two new variables: an "Age" variable based on the firm's IPO date and an "Interest Coverage Ratio" variable equal to EBIT divided by interest expense. She also deletes the "IPO Date" variable from the dataset. After applying these transformations, Steele scales the financial data using normalization. She notes that over the full sample dataset, the "Interest Expense" variable ranges from a minimum of 0.2

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and a maximum of 12.2, with a mean of 1.1 and a standard deviation of 0.4.

Steele and Schultz then discuss how to preprocess the raw text data. Steele tells Schultz that the process can be completed in the following three steps:

Step 1 Cleanse the raw text data.

Step 2 Split the cleansed data into a collection of words for them to be normalized.

Step 3 Normalize the collection of words from Step 2 and create a distinct set of tokens from the normalized words.

With respect to Step 1, Steele tells Schultz:

"I believe I should remove all html tags, punctuations, numbers, and extra white spaces from the data before normalizing them."

After properly cleansing the raw text data, Steele completes Steps 2 and 3. She then performs exploratory data analysis. To assist in feature selection, she wants to create a visualization that shows the most informative words in the dataset based on their term frequency (TF) values. After creating and analyzing the visualization, Steele is concerned that some tokens are likely to be noise features for ML model training; therefore, she wants to remove them.

Steele and Schultz discuss the importance of feature selection and feature engineering in ML model training. Steele tells Schultz:

"Appropriate feature selection is a key factor in minimizing model overfitting, whereas feature engineering tends to prevent model underfitting."

Once satisfied with the final set of features, Steele selects and runs a model on the training set that classifies the text as having positive sentiment (Class "1" or negative sentiment (Class "0"). She then evaluates its performance using error analysis. The resulting confusion matrix is presented in Exhibit 2.

Exhibit 2: Confusion Matrix

		Actual Training Results	
		Class "1"	Class "0"
Predicted Results	Class "1"	TP = 182	FP = 52
	Class "0"	FN = 31	TN = 96

1. Which of Steele's statements relating to the steps in building structured data-based and text-based ML models is correct?

   A. Only Statement 1 is correct.

   B. Only Statement 2 is correct.

   C. Statement 1 and Statement 2 are correct.

2. Steele's concern about using Twitter data in the model *best* relates to:

   A. volume.

   B. velocity.

   C. veracity.

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3. What type of error appears to be present in the IPO Date column of Exhibit 1?
   A. invalidity error.
   B. inconsistency error.
   C. non-uniformity error.

4. What type of error is most likely present in the last row of data (ID #4) in Exhibit 1?
   A. Inconsistency error
   B. Incompleteness error
   C. Non-uniformity error

5. During the preprocessing of the data in Exhibit 1, what type of data transformation did Steele perform during the data preprocessing step?
   A. Extraction
   B. Conversion
   C. Aggregation

6. Based on Exhibit 1, for the firm with ID #3, Steele should compute the scaled value for the "Interest Expense" variable as:
   A. 0.008.
   B. 0.083.
   C. 0.250.

7. Is Steele's statement regarding Step 1 of the preprocessing of raw text data correct?
   A. Yes.
   B. No, because her suggested treatment of punctuation is incorrect.
   C. No, because her suggested treatment of extra white spaces is incorrect.

8. Steele's Step 2 can be *best* described as:
   A. tokenization.
   B. lemmatization.
   C. standardization.

9. The output created in Steele's Step 3 can be *best* described as a:
   A. bag-of-words.
   B. set of n-grams.
   C. document term matrix.

10. Given her objective, the visualization that Steele should create in the exploratory

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data analysis step is a:

A. scatter plot.

B. word cloud.

C. document term matrix.

11. To address her concern in her exploratory data analysis, Steele should focus on those tokens that have:

A. low chi-square statistics.

B. low mutual information (ML) values.

C. very low and very high term frequency (TF) values.

12. Is Steele's statement regarding the relationship between feature selection/feature engineering and model fit correct?

A. Yes.

B. No, because she is incorrect with respect to feature selection.

C. No, because she is incorrect with respect to feature engineering.

13. Based on Exhibit 2, the model's precision metric is *closest* to:

A. 78%.

B. 81%.

C. 85%.

14. Based on Exhibit 2, the model's F1 score is *closest* to:

A. 77%.

B. 81%.

C. 85%.

15. Based on Exhibit 2, the model's accuracy metric is *closest* to:

A. 77%.

B. 81%.

C. 85%.

The following information relates to questions 16-24

Bernadette Rivera is a portfolio manager at Voxkor, a private equity company that provides financing to early-stage start-up businesses. Rivera is working with a data analyst, Tim Achler, on a text-based machine-learning (ML) model to enhance Voxkor's predictive ability to identify successful start-ups.

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Voxkor currently uses ML models based only on traditional, structured financial data but would like to develop a new ML model that analyzes textual big data gathered from the internet. The model will classify text information into positive or negative sentiment classes for each respective start-up. Rivera wants to confirm her understanding of any differences in the ML model building steps between data analysis projects that use traditional structured data and projects that involve unstructured, text-based data. Rivera makes the following statements:

Statement 1 Some of the methods used in the exploration step are different for structured and unstructured data, but for both types of data, the step involves feature selection and feature engineering.

Statement 2 A major difference when developing a text-based ML model is the curation step, which involves cleansing, preprocessing, and converting the data into a structured format usable for model training.

Achler uses a web spidering program to obtain the data for the text-based model. The program extracts raw content from social media webpages, which contains English language sentences and special characters. After curating the text, Achler removes unnecessary elements from the raw text using regular expression software and completes additional text cleansing and preprocessing tasks.

Next, Achler and Rivera discuss remaining text wrangling tasks—specifically, which tokens to include in the document term matrix (DTM). Achler divides unique tokens into three groups; a sample of each group is shown in Exhibit 1.

Exhibit 1: Summary of Sample Tokens

Token Group 1	Token Group 2	Token Group 3
"not_increas_market"	"not_increased_market"	"not," "increased," "market"
"currencysign"	"currencysign"	"EUR"
"sale_decreas"	"sale_decreased"	"Sales," "decreased"

The dataset is now ready for the text exploration step. At this point in the process, Rivera wants to better comprehend the collection of unique words. Achler recommends an exploratory data analysis technique that visualizes words by varying their font size proportionately to the number of occurrences of each word in the corpus.

As an additional part of the text exploration step, Achler conducts a term frequency analysis to identify outliers. Achler summarizes the analysis in Exhibit 2.

Exhibit 2: Words with Highest and Lowest Frequency Value

Group 1		Group 2	
Word	**Frequency**	**Word**	**Frequency**
the	0.04935	naval	1.0123e-05
and	0.04661	stereotype	1.5185e-05
to	0.04179	till	1.5185e-05

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Group 1		Group 2	
Word	**Frequency**	**Word**	**Frequency**
that	0.03577	ribbon	2.0247e-05
in	0.03368	deposit	2.5308e-05

Note: "e-05" represents 10^{-5}.

Achler has the data ready for the model training process. Rivera asks Achler to include start-up failure rates as a feature. Achler notices that the number of start-ups that fail (majority class) is significantly larger than the number of the start-ups that are successful (minority class). Achler is concerned that because of class imbalance, the model will not be able to discriminate between start-ups that fail and start-ups that are successful.

Achler splits the DTM into training, cross-validation, and test datasets. Achler uses a supervised learning approach to train the logistic regression model in predicting sentiment. Applying the receiver operating characteristics (ROC) technique and area under the curve (AUC) metrics, Achler evaluates model performance on both the training and the cross-validation datasets. The trained model performance for three different logistic regressions' threshold *p*-values is presented in Exhibit 3.

Exhibit 3: AUC for Different Threshold p-values

Threshold *p*-Value	Training Set	Cross-Validation Set
$p = 0.57$	56.7%	57.3%
$p = 0.79$	91.3%	89.7%
$p = 0.84$	98.4%	87.1%

Rivera suggests adjusting the model's hyperparameters to improve performance. Achler runs a grid search that compares the difference between the prediction error on both the training and the cross-validation datasets for various combinations of hyperparameter values. For the current values of hyperparameters, Achler observes that the prediction error on the training dataset is small, whereas the prediction error on the cross-validation dataset is significantly larger.

16. Which of Rivera's statements about differences in ML model building steps is correct?

A. Only Statement 1

B. Only Statement 2

C. Both Statement 1 and Statement 2

17. Based on the source of the data, as part of the data cleansing and wrangling process, Achler *most likely* needs to remove:

A. html tags and perform scaling.

B. numbers and perform lemmatization.

C. white spaces and perform winsorization.

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18. Based on Exhibit 1, which token group has *most likely* undergone the text preparation and wrangling process?

A. Token Group 1

B. Token Group 2

C. Token Group 3

19. The visual text representation technique that Achler recommends to Rivera is a:

A. word cloud.

B. bag of words.

C. collection frequency.

20. Based on Exhibit 2, Achler should exclude from further analysis words in:

A. only Group 1.

B. only Group 2.

C. both Group 1 and Group 2.

21. Achler's model training concern related to the model's ability to discriminate could be addressed by randomly:

A. oversampling the failed start-up data.

B. oversampling the successful start-up data.

C. undersampling the successful start-up data.

22. Based on Exhibit 3, which threshold *p*-value indicates the *best* fitting model?

A. 0.57

B. 0.79

C. 0.84

23. Based on Exhibit 3, if Achler wants to improve model performance at the threshold *p*-value of 0.84, he should:

A. tune the model to lower the AUC.

B. adjust model parameters to decrease ROC convexity.

C. apply LASSO regularization to the logistic regression.

24. Based on Achler's grid search analysis, the current model can be characterized as:

A. underfitted.

B. having low variance.

C. exhibiting slight regularization.

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The following information relates to questions 25-31

Iesha Azarov is a senior analyst at Ganymede Moon Partners (Ganymede), where he works with junior analyst Pàola Bector. Azarov would like to incorporate machine learning (ML) models into the company's analytical process. Azarov asks Bector to develop ML models for two unstructured stock sentiment datasets, Dataset ABC and Dataset XYZ. Both datasets have been cleaned and preprocessed in preparation for text exploration and model training.

Following an exploratory data analysis that revealed Dataset ABC's most frequent tokens, Bector conducts a collection frequency analysis. Bector then computes TF–IDF (term frequency–inverse document frequency) for several words in the collection and tells Azarov the following:

Statement 1 IDF is equal to the inverse of the document frequency measure.

Statement 2 TF at the collection level is multiplied by IDF to calculate TF–IDF.

Statement 3 TF–IDF values vary by the number of documents in the dataset, and therefore, model performance can vary when applied to a dataset with just a few documents.

Bector notes that Dataset ABC is characterized by the absence of ground truth.

Bector turns his attention to Dataset XYZ, containing 84,000 tokens and 10,000 sentences. Bector chooses an appropriate feature selection method to identify and remove unnecessary tokens from the dataset and then focuses on model training. For performance evaluation purposes, Dataset XYZ is split into a training set, cross-validation (CV) set, and test set. Each of the sentences has already been labeled as either a positive sentiment (Class "1") or a negative sentiment (Class "0") sentence. There is an unequal class distribution between the positive sentiment and negative sentiment sentences in Dataset XYZ. Simple random sampling is applied within levels of the sentiment class labels to balance the class distributions within the splits. Bector's view is that the false positive and false negative evaluation metrics should be given equal weight. Select performance data from the cross-validation set confusion matrices is presented in Exhibit 1:

Exhibit 1: Performance Metrics for Dataset XYZ

Confusion Matrix	CV Data (threshold *p*-value)	Performance Metrics			
		Precision	Recall	F1 Score	Accuracy
A	0.50	0.95	0.87	0.91	0.91
B	0.35	0.93	0.90	0.91	0.92
C	0.65	0.86	0.97	0.92	0.91

Azarov and Bector evaluate the Dataset XYZ performance metrics for Confusion Matrices A, B, and C in Exhibit 1. Azarov says, "For Ganymede's purposes, we should be most concerned with the cost of Type I errors."

Azarov requests that Bector apply the ML model to the test dataset for Dataset XYZ, assuming a threshold *p*-value of 0.65. Exhibit 2 contains a sample of results from the test dataset corpus.

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Exhibit 2: 10 Sample Results of Test Data for Dataset XYZ

Sentence #	Actual Sentiment	Target *p*-Value
1	1	0.75
2	0	0.45
3	1	0.64
4	1	0.81
5	0	0.43
6	1	0.78
7	0	0.59
8	1	0.60
9	0	0.67
10	0	0.54

Bector makes the following remarks regarding model training:

Remark 1 Method selection is governed by such factors as the type of data and the size of data.

Remark 2 In the performance evaluation stage, model fitting errors, such as bias error and variance error, are used to measure goodness of fit.

25. Based on the text exploration method used for Dataset ABC, tokens that potentially carry important information useful for differentiating the sentiment embedded in the text are *most likely* to have values that are:

A. low.

B. intermediate.

C. high.

26. Which of Bector's statements regarding TF, IDF, and TF–IDF is correct?

A. Statement 1

B. Statement 2

C. Statement 3

27. What percentage of Dataset ABC should be allocated to a training subset?

A. 0%

B. 20%

C. 60%

28. Based only on Dataset XYZ's composition and Bector's view regarding false positive and false negative evaluation metrics, which performance measure is *most appropriate*?

A. Recall

B. F1 score

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C. Precision

29. Based on Exhibit 1, which confusion matrix demonstrates the *most* favorable value of the performance metric that *best* addresses Azarov's concern?

A. Confusion Matrix A

B. Confusion Matrix B

C. Confusion Matrix C

30. Based on Exhibit 2, the accuracy metric for Dataset XYZ's test set sample is *closest to*:

A. 0.67.

B. 0.70.

C. 0.75.

31. Which of Bector's remarks related to model training is correct?

A. Only Remark 1

B. Only Remark 2

C. Both Remark 1 and Remark 2

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SOLUTIONS

1. B is correct. The five steps in building structured data-based ML models are: 1) conceptualization of the modeling task, 2) data collection, 3) data preparation and wrangling, 4) data exploration, and 5) model training. The five steps in building text-based ML models are: 1) text problem formulation, 2) data (text) curation, 3) text preparation and wrangling, 4) text exploration, and 5) model training. Statement 1 is incorrect: Text preparation and wrangling is the third step in building text ML models and occurs after the second data (text) curation step. Statement 2 is correct: The fourth step in building both types of models encompasses data/text exploration.

2. C is correct. Veracity relates to the credibility and reliability of different data sources. Steele is concerned about the credibility and reliability of Twitter content, noting that research suggests that as much as 10%–15% of social media content is from fake accounts.

3. C is correct. A non-uniformity error occurs when the data are not presented in an identical format. The data in the "IPO Date" column represent the IPO date of each firm. While all rows are populated with valid dates in the IPO Date column, the dates are presented in different formats (e.g., mm/dd/yyyy, dd/mm/yyyy).

4. A is correct. There appears to be an inconsistency error in the last row (ID #4). An inconsistency error occurs when a data point conflicts with corresponding data points or reality. In the last row, the interest expense data item has a value of 1.5, and the total debt item has a value of 0.0. This appears to be an error: Firms that have interest expense are likely to have debt in their capital structure, so either the interest expense is incorrect or the total debt value is incorrect. Steele should investigate this issue by using alternative data sources to confirm the correct values for these variables.

5. A is correct. During the data preprocessing step, Steele created a new "Age" variable based on the firm's IPO date and then deleted the "IPO Date" variable from the dataset. She also created a new "Interest Coverage Ratio" variable equal to EBIT divided by interest expense. Extraction refers to a data transformation where a new variable is extracted from a current variable for ease of analyzing and using for training an ML model, such as creating an age variable from a date variable or a ratio variable. Steele also performed a selection transformation by deleting the IPO Date variable, which refers to deleting the data columns that are not needed for the project.

6. B is correct. Steele uses normalization to scale the financial data. Normalization is the process of rescaling numeric variables in the range of [0, 1]. To normalize variable X, the minimum value (X_{min}) is subtracted from each observation (X_i), and then this value is divided by the difference between the maximum and minimum values of X ($X_{\text{max}} - X_{\text{min}}$):

   $$X_{i\ (\text{normalized})} = \frac{X_i - X_{\text{min}}}{X_{\text{max}} - X_{\text{min}}}.$$

   The firm with ID #3 has an interest expense of 1.2. So, its normalized value is calculated as:

   $$X_{i\ (\text{normalized})} = \frac{1.2 - 0.2}{12.2 - 0.2} = 0.083.$$

7. B is correct. Although most punctuations are not necessary for text analysis and

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should be removed, some punctuations (e.g., percentage signs, currency symbols, and question marks) may be useful for ML model training. Such punctuations should be substituted with annotations (e.g., /percentSign/, /dollarSign/, and /questionMark/) to preserve their grammatical meaning in the text. Such annotations preserve the semantic meaning of important characters in the text for further text processing and analysis stages.

8. A is correct. Tokenization is the process of splitting a given text into separate tokens. This step takes place after cleansing the raw text data (removing html tags, numbers, extra white spaces, etc.). The tokens are then normalized to create the bag-of-words (BOW).

9. A is correct. After the cleansed text is normalized, a bag-of-words is created. A bag-of-words (BOW) is a collection of a distinct set of tokens from all the texts in a sample dataset.

10. B is correct. Steele wants to create a visualization for Schultz that shows the most informative words in the dataset based on their term frequency (TF, the ratio of the number of times a given token occurs in the dataset to the total number of tokens in the dataset) values. A word cloud is a common visualization when working with text data as it can be made to visualize the most informative words and their TF values. The most commonly occurring words in the dataset can be shown by varying font size, and color is used to add more dimensions, such as frequency and length of words.

11. C is correct. Frequency measures can be used for vocabulary pruning to remove noise features by filtering the tokens with very high and low TF values across all the texts. Noise features are both the most frequent and most sparse (or rare) tokens in the dataset. On one end, noise features can be stop words that are typically present frequently in all the texts across the dataset. On the other end, noise features can be sparse terms that are present in only a few text files. Text classification involves dividing text documents into assigned classes. The frequent tokens strain the ML model to choose a decision boundary among the texts as the terms are present across all the texts (an example of underfitting). The rare tokens mislead the ML model into classifying texts containing the rare terms into a specific class (an example of overfitting). Thus, identifying and removing noise features are critical steps for text classification applications.

12. A is correct. A dataset with a small number of features may not carry all the characteristics that explain relationships between the target variable and the features. Conversely, a large number of features can complicate the model and potentially distort patterns in the data due to low degrees of freedom, causing overfitting. Therefore, appropriate feature selection is a key factor in minimizing such model overfitting. Feature engineering tends to prevent underfitting in the training of the model. New features, when engineered properly, can elevate the underlying data points that better explain the interactions of features. Thus, feature engineering can be critical to overcome underfitting.

13. A is correct. Precision, the ratio of correctly predicted positive classes (true positives) to all predicted positive classes, is calculated as:

    Precision (P) = TP/(TP + FP) = 182/(182 + 52) = 0.7778 (78%).

14. B is correct. The model's F1 score, which is the harmonic mean of precision and recall, is calculated as:

    F1 score = (2 × P × R)/(P + R).

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F1 score = (2 × 0.7778 × 0.8545)/(0.7778 + 0.8545) = 0.8143 (81%).

15. A is correct. The model's accuracy, which is the percentage of correctly predicted classes out of total predictions, is calculated as:

    Accuracy = (TP + TN)/(TP + FP + TN + FN).

    Accuracy = (182 + 96)/(182 + 52 + 96 + 31) = 0.7701 (77%).

16. A is correct. Statement 1 is correct because some of the methods used in the fourth step of ML model building (data/text exploration) are different for structured and unstructured data, and for both structured and unstructured data, the exploration step encompasses feature selection and feature engineering. Statement 2 is incorrect because Rivera described the text preparation and wrangling step, not the text curation step. The data (text) curation step involves gathering relevant external text data via web services or programs that extract raw content from a source.

    B and C are incorrect because Statement 2 is incorrect. Rivera described the text preparation and wrangling step, not the text curation step. The data (text) curation step involves gathering relevant external text data via web services or programs that extract raw content from a source.

17. B is correct. Achler uses a web spidering program that extracts unstructured raw content from social media webpages. Raw text data are a sequence of characters and contain other non-useful elements including html tags, punctuation, and white spaces (including tabs, line breaks, and new lines). Removing numbers is one of the basic operations in the text cleansing/preparation process for unstructured data. When numbers (or digits) are present in the text, they should be removed or substituted with the annotation "/number/." Lemmatization, which takes places during the text wrangling/preprocessing process for unstructured data, is the process of converting inflected forms of a word into its morphological root (known as lemma). Lemmatization reduces the repetition of words occurring in various forms while maintaining the semantic structure of the text data, thereby aiding in training less complex ML models.

    A is incorrect because although html tag removal is part of text cleansing/preparation for unstructured data, scaling is a data wrangling/preprocessing process applied to structured data. Scaling adjusts the range of a feature by shifting and changing the scale of data; it is performed on numeric variables, not on text data.

    C is incorrect because although raw text contains white spaces (including tabs, line breaks, and new lines) that need to be removed as part of the data cleansing/preparation process for unstructured data, winsorization is a data wrangling/preprocessing task performed on values of data points, not on text data. Winsorization is used for structured numerical data and replaces extreme values and outliers with the maximum (for large-value outliers) and minimum (for small-value outliers) values of data points that are not outliers.

18. A is correct. Data preparation and wrangling involve cleansing and organizing raw data into a consolidated format. Token Group 1 includes n-grams ("not_increas_market," "sale_decreas") and the words that have been converted from their inflected forms into their base word ("increas," "decreas"), and the currency symbol has been replaced with a "currencysign" token. N-gram tokens are helpful for keeping negations intact in the text, which is vital for sentiment prediction. The process of converting inflected forms of a word into its base word is called stemming and helps decrease data sparseness, thereby aiding in training less

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complex ML models.

B is incorrect because Token Group 2 includes inflected forms of words ("increased," "decreased") before conversion into their base words (known as stems). Stemming (along with lemmatization) decreases data sparseness by aggregating many sparsely occurring words in relatively less sparse stems or lemmas, thereby aiding in training less complex ML models.

C is incorrect because Token Group 3 includes inflected forms of words ("increased," "decreased") before conversion into their base words (known as stems). In addition, the "EUR" currency symbol has not been replaced with the "currencysign" token and the word "Sales" has not been lowercased.

19. A is correct. Achler recommends creating a word cloud, which is a common text visualization technique at the data exploration phase in ML model building. The most commonly occurring words in the dataset can be visualized by varying font size, and color is used to add more dimensions, such as frequency and length of words.

    B is incorrect because Achler recommends creating a word cloud and not a bag of words (BOW). A BOW is a collection of a distinct set of tokens from all the texts in a sample dataset. A BOW representation is a basic procedure used primarily to analyze text during Step 3 (text wrangling/preprocessing), although it may also be used in Step 4 during the feature engineering process. In contrast to a word cloud, which visually varies font size and color, BOW is simply a set of words (typically displayed in table).

    C is incorrect because Achler recommends creating a word cloud and not a collection frequency. Collection frequency (or term frequency) is the ratio of the number of times a given token occurs in all the texts in the dataset to the total number of tokens in the dataset. Collection frequency can be calculated and examined to identify outlier words, but it is not a visual text representation tool.

20. C is correct. Achler should remove words that are in both Group 1 and Group 2. Term frequency values range between 0 and 1. Group 1 consists of the highest frequency values (e.g., "the" = 0.04935), and Group 2 consists of the lowest frequency values (e.g., "naval" = 1.0123e–05). Frequency analysis on the processed text data helps in filtering unnecessary tokens (or features) by quantifying how important tokens are in a sentence and in the corpus as a whole. The most frequent tokens (Group 1) strain the machine-learning model to choose a decision boundary among the texts as the terms are present across all the texts, which leads to model underfitting. The least frequent tokens (Group 2) mislead the machine-learning model into classifying texts containing the rare terms into a specific class, which leads to model overfitting. Identifying and removing noise features is critical for text classification applications.

    A is incorrect because words in both Group 1 and Group 2 should be removed. The words with high term frequency value are mostly stop words, present in most sentences. Stop words do not carry a semantic meaning for the purpose of text analyses and ML training, so they do not contribute to differentiating sentiment.

    B is incorrect because words in both Group 1 and Group 2 should be removed. Terms with low term frequency value are mostly rare terms, ones appearing only once or twice in the data. They do not contribute to differentiating sentiment.

21. B is correct. Achler is concerned about class imbalance, which can be resolved by balancing the training data. The majority class (the failed start-up data) can be randomly undersampled, or the minority class (the successful start-up data) can be randomly oversampled.

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22. B is correct. The higher the AUC, the better the model performance. For the threshold *p*-value of 0.79, the AUC is 91.3% on the training dataset and 89.7% on the cross-validation dataset, and the ROC curves are similar for model performance on both datasets. These findings suggest that the model performs similarly on both training and CV data and thus indicate a good fitting model.

    A is incorrect because for the threshold *p*-value of 0.57, the AUC is 56.7% on the training dataset and 57.3% on the cross-validation dataset. The AUC close to 50% signifies random guessing on both the training dataset and the cross-validation dataset. The implication is that for the threshold *p*-value of 0.57, the model is randomly guessing and is not performing well.

    C is incorrect because for the threshold *p*-value of 0.84, there is a substantial difference between the AUC on the training dataset (98.4%) and the AUC on the cross-validation dataset (87.1%). This suggests that the model performs comparatively poorly (with a higher rate of error or misclassification) on the cross-validation dataset when compared with training data. Thus, the implication is that the model is overfitted.

23. C is correct. At the threshold *p*-value of 0.84, the AUC is 98.4% for the training dataset and 87.1% for the cross-validation dataset, which suggests that the model is currently overfitted. Least absolute shrinkage and selection operator (LASSO) regularization can be applied to the logistic regression to prevent overfitting of logistic regression models.

    A is incorrect because the higher the AUC, the better the model performance.

    B is incorrect because the more convex the ROC curve and the higher the AUC, the better the model performance. Adjusting model parameters with the aim of achieving lower ROC convexity would result in worse model performance on the cross-validation dataset.

24. C is correct. Slight regularization occurs when the prediction error on the training dataset is small, while the prediction error on the cross-validation data set is significantly larger. This difference in error is variance. High variance error, which typically is due to too many features and model complexity, results in model overfitting.

    A is incorrect. The current model has high variance which results in model overfitting, not underfitting.

    B is incorrect. The difference between the prediction error on the training dataset and the prediction error on the cross-validation dataset is high, which means that the current model has high variance, not low.

25. B is correct. When analyzing term frequency at the corpus level, also known as collection frequency, tokens with intermediate term frequency (TF) values potentially carry important information useful for differentiating the sentiment embedded in the text. Tokens with the highest TF values are mostly stop words that do not contribute to differentiating the sentiment embedded in the text, and tokens with the lowest TF values are mostly proper nouns or sparse terms that are also not important to the meaning of the text.

    A is incorrect because tokens with the lowest TF values are mostly proper nouns or sparse terms (noisy terms) that are not important to the meaning of the text.

    C is incorrect because tokens with the highest TF values are mostly stop words (noisy terms) that do not contribute to differentiating the sentiment embedded in the text.

26. C is correct. Statement 3 is correct. TF–IDF values vary by the number of documents in the dataset, and therefore, the model performance can vary when

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applied to a dataset with just a few documents.

Statement 1 is incorrect because IDF is calculated as the log of the inverse, or reciprocal, of the document frequency measure. Statement 2 is incorrect because TF at the sentence (not collection) level is multiplied by IDF to calculate TF–IDF.

A is incorrect because Statement 1 is incorrect. IDF is calculated as the log of the inverse, or reciprocal, of the document frequency (DF) measure.

B is incorrect because Statement 2 is incorrect. TF at the sentence (not collection) level is multiplied by IDF to calculate TF–IDF.

27. A is correct; 0% of the master dataset of Dataset ABC should be allocated to a training subset. Dataset ABC is characterized by the absence of ground truth (i.e., no known outcome or target variable) and is therefore an unsupervised ML model. For unsupervised learning models, no splitting of the master dataset is needed, because of the absence of labeled training data. Supervised ML datasets (with labeled training data) contain ground truth, the known outcome (target variable) of each observation in the dataset.

    B is incorrect because 20% is the commonly recommended split for the cross-validation set and test set in supervised training ML datasets. Dataset ABC is an unsupervised ML dataset, for which no splitting (0%) of the master dataset is needed, because of the absence of labeled training data. In supervised ML models (which contain labeled training data), the master dataset is split into three subsets (a training set, cross-validation set, and test set), which are used for model training and testing purposes.

    C is incorrect because 60% is the commonly recommended split for the training set in supervised training ML datasets. Dataset ABC is an unsupervised ML dataset, for which no splitting (0%) of the master dataset is needed, because of the absence of labeled training data. In supervised ML models (which contain labeled training data), the master dataset is split into three subsets (a training set, cross-validation set, and test set), which are used for model training and testing purposes.

28. B is correct. F1 score is the most appropriate performance measure for Dataset XYZ. Bector gives equal weight to false positives and false negatives. Accuracy and F1 score are overall performance measures that give equal weight to false positives and false negatives. Accuracy is considered an appropriate performance measure for balanced datasets, where the number of "1" and "0" classes are equal. F1 score is considered more appropriate than accuracy when there is unequal class distribution in the dataset and it is necessary to measure the equilibrium of precision and recall. Since Dataset XYZ contains an unequal class distribution between positive and negative sentiment sentences, F1 score is the most appropriate performance measure.

    Precision is the ratio of correctly predicted positive classes to all predicted positive classes and is useful in situations where the cost of false positives or Type I errors is high. Recall is the ratio of correctly predicted positive classes to all actual positive classes and is useful in situations where the cost of false negatives or Type II errors is high.

    A is incorrect because Bector gives equal weight to false positives and false negatives. Accuracy and F1 score are overall performance measures that give equal weight to false positives and false negatives. Recall is the ratio of correctly predicted positive classes to all actual positive classes and is useful in situations where the cost of false negatives or Type II errors is high.

    C is incorrect because Bector gives equal weight to false positive and false negatives. Accuracy and F1 score are overall performance measures that give equal weight to false positives and false negatives. Precision is the ratio of correctly predicted positive classes to all predicted positive classes and is useful in situa-

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tions where the cost of false positives or Type-I error is high.

29. A is correct. Precision is the ratio of correctly predicted positive classes to all predicted positive classes and is useful in situations where the cost of false positives or Type I errors is high. Confusion Matrix A has the highest precision and therefore demonstrates the most favorable value of the performance metric that best addresses Azarov's concern about the cost of Type I errors. Confusion Matrix A has a precision score of 0.95, which is higher than the precision scores of Confusion Matrix B (0.93) and Confusion Matrix C (0.86).

    B is incorrect because precision, not accuracy, is the performance measure that best addresses Azarov's concern about the cost of Type I errors. Confusion Matrix B demonstrates the most favorable value for the accuracy score (0.92), which is higher than the accuracy scores of Confusion Matrix A (0.91) and Confusion Matrix C (0.91). Accuracy is a performance measure that gives equal weight to false positives and false negatives and is considered an appropriate performance measure when the class distribution in the dataset is equal (a balanced dataset). However, Azarov is most concerned with the cost of false positives, or Type I errors, and not with finding the equilibrium between precision and recall. Furthermore, Dataset XYZ has an unequal (unbalanced) class distribution between positive sentiment and negative sentiment sentences.

    C is incorrect because precision, not recall or F1 score, is the performance measure that best addresses Azarov's concern about the cost of Type I errors. Confusion Matrix C demonstrates the most favorable value for the recall score (0.97), which is higher than the recall scores of Confusion Matrix A (0.87) and Confusion Matrix B (0.90). Recall is the ratio of correctly predicted positive classes to all actual positive classes and is useful in situations where the cost of false negatives, or Type II errors, is high. However, Azarov is most concerned with the cost of Type I errors, not Type II errors.

    F1 score is more appropriate (than accuracy) when there is unequal class distribution in the dataset and it is necessary to measure the equilibrium of precision and recall. Confusion Matrix C demonstrates the most favorable value for the F1 score (0.92), which is higher than the F1 scores of Confusion Matrix A (0.91) and Confusion Matrix B (0.91). Although Dataset XYZ has an unequal class distribution between positive sentiment and negative sentiment sentences, Azarov is most concerned with the cost of false positives, or Type I errors, and not with finding the equilibrium between precision and recall.

30. B is correct. Accuracy is the percentage of correctly predicted classes out of total predictions and is calculated as (TP + TN)/(TP + FP + TN + FN).

    In order to obtain the values for true positive (TP), true negative (TN), false positive (FP), and false negative (FN), predicted sentiment for the positive (Class "1") and the negative (Class "0") classes are determined based on whether each individual target *p*-value is greater than or less than the threshold *p*-value of 0.65. If an individual target *p*-value is greater than the threshold *p*-value of 0.65, the predicted sentiment for that instance is positive (Class "1"). If an individual target *p*-value is less than the threshold *p*-value of 0.65, the predicted sentiment for that instance is negative (Class "0"). Actual sentiment and predicted sentiment are then classified as follows:

Actual Sentiment	Predicted Sentiment	Classification
1	1	TP
0	1	FP

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Actual Sentiment	Predicted Sentiment	Classification
1	0	FN
0	0	TN

Exhibit 2, with added "Predicted Sentiment" and "Classification" columns, is presented below:

10 Sample Results of Test Data for Dataset XYZ

Sentence #	Actual Sentiment	Target *p*-Value	Predicted Sentiment	Classification
1	1	0.75	1	TP
2	0	0.45	0	TN
3	1	0.64	0	FN
4	1	0.81	1	TP
5	0	0.43	0	TN
6	1	0.78	1	TP
7	0	0.59	0	TN
8	1	0.60	0	FN
9	0	0.67	1	FP
10	0	0.54	0	TN

Based on the classification data obtained from Exhibit 2, a confusion matrix can be generated:

Confusion Matrix for Dataset XYZ Sample Test Data with Threshold *p*-Value = 0.65

	Actual Training Labels	
Predicted Results	Class "1"	Class "0"
Class "1"	TP = 3	FP = 1
Class "0"	FN = 2	TN = 4

Using the data in the confusion matrix above, the accuracy metric is computed as follows:

Accuracy = (TP + TN)/(TP + FP + TN + FN).

Accuracy = (3 + 4)/(3 + 1 + 4 + 2) = 0.70.

A is incorrect because 0.67 is the F1 score, not accuracy metric, for the sample of the test set for Dataset XYZ, based on Exhibit 2. To calculate the F1 score, the precision (P) and the recall (R) ratios must first be calculated. Precision and recall for the sample of the test set for Dataset XYZ, based on Exhibit 2, are calculated as follows:

Precision (P) = TP/(TP + FP) = 3/(3 + 1) = 0.75.

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Recall (R) = TP/(TP + FN) = 3/(3 + 2) = 0.60.

The F1 score is calculated as follows:

F1 score = (2 × P × R)/(P + R) = (2 × 0.75 × 0.60)/(0.75 + 0.60)

= 0.667, or 0.67.

C is incorrect because 0.75 is the precision ratio, not the accuracy metric, for the sample of the test set for Dataset XYZ, based on Exhibit 2. The precision score is calculated as follows:

Precision (P) = TP/(TP + FP) = 3/(3 + 1) = 0.75.

31. A is correct. Only Remark 1 is correct. Method selection is the first task of ML model training and is governed by the following factors: (1) supervised or unsupervised learning, (2) the type of data, and (3) the size of data. The second and third tasks of model training, respectively, are performance evaluation and tuning.

Remark 2 is incorrect because model fitting errors (bias error and variance error) are used in tuning, not performance evaluation. The techniques used in performance evaluation, which measure the goodness of fit for validation of the model, include (1) error analysis, (2) receiver operating characteristic (ROC) plots, and (3) root mean squared error (RMSE) calculations.

B and C are incorrect because Remark 2 is incorrect. Model fitting errors (bias error and variance error) are used in tuning, not performance evaluation. The techniques used in performance evaluation, which measure the goodness of fit for validation of the model, include (1) error analysis, (2) receiver operating characteristic plots, and (3) root mean squared error calculations.

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LEARNING MODULE

8

Appendices A-E

APPENDICES A-E

1

Appendix A	Cumulative Probabilities for a Standard Normal Distribution
Appendix B	Table of the Student's *t*-Distribution (One-Tailed Probabilities)
Appendix C	Values of X^2 (Degrees of Freedom, Level of Significance)
Appendix D	Table of the *F*-Distribution
Appendix E	Critical Values for the Durbin-Watson Statistic (α = .05)

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Appendix A
Cumulative Probabilities for a Standard Normal Distribution
$P(Z \leq x) = N(x)$ for $x \geq 0$ or $P(Z \leq z) = N(z)$ for $z \geq 0$

x or z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.00	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.10	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.20	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.30	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.40	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.50	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.60	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.70	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.80	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.90	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.00	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.10	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.20	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.30	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.40	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.50	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.60	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.70	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.80	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.90	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.00	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.10	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.20	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.30	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.40	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.50	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.60	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.70	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.80	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.90	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.00	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.10	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.20	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.30	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.40	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.50	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.60	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.70	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.80	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.90	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4.00	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

For example, to find the *z*-value leaving 2.5 percent of the area/probability in the upper tail, find the element 0.9750 in the body of the table. Read 1.90 at the left end of the element's row and 0.06 at the top of the element's column, to give 1.90 + 0.06 = 1.96. *Table generated with Excel.*

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Appendix A (continued)
Cumulative Probabilities for a Standard Normal Distribution
$P(Z \leq x) = N(x)$ for $x \leq 0$ or $P(Z \leq z) = N(z)$ for $z \leq 0$

x or z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
−0.10	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.20	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.30	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.40	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.50	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.60	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.70	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.80	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.90	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−1.00	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−1.10	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.20	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.30	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.40	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.50	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.60	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.70	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.80	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.90	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−2.00	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−2.10	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.20	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.30	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.40	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.50	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.60	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.70	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.80	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.90	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−3.00	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−3.10	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.20	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.30	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.40	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.50	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002
−3.60	0.0002	0.0002	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
−3.70	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
−3.80	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
−3.90	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
−4.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

For example, to find the z-value leaving 2.5 percent of the area/probability in the lower tail, find the element 0.0250 in the body of the table. Read −1.90 at the left end of the element's row and 0.06 at the top of the element's column, to give −1.90 − 0.06 = −1.96. *Table generated with Excel.*

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Appendix B
Table of the Student's *t*-Distribution (One-Tailed Probabilities)

df	*p* = 0.10	*p* = 0.05	*p* = 0.025	*p* = 0.01	*p* = 0.005	df	*p* = 0.10	*p* = 0.05	*p* = 0.025	*p* = 0.01	*p* = 0.005
1	3.078	6.314	12.706	31.821	63.657	**31**	1.309	1.696	2.040	2.453	2.744
2	1.886	2.920	4.303	6.965	9.925	**32**	1.309	1.694	2.037	2.449	2.738
3	1.638	2.353	3.182	4.541	5.841	**33**	1.308	1.692	2.035	2.445	2.733
4	1.533	2.132	2.776	3.747	4.604	**34**	1.307	1.691	2.032	2.441	2.728
5	1.476	2.015	2.571	3.365	4.032	**35**	1.306	1.690	2.030	2.438	2.724
6	1.440	1.943	2.447	3.143	3.707	**36**	1.306	1.688	2.028	2.434	2.719
7	1.415	1.895	2.365	2.998	3.499	**37**	1.305	1.687	2.026	2.431	2.715
8	1.397	1.860	2.306	2.896	3.355	**38**	1.304	1.686	2.024	2.429	2.712
9	1.383	1.833	2.262	2.821	3.250	**39**	1.304	1.685	2.023	2.426	2.708
10	1.372	1.812	2.228	2.764	3.169	**40**	1.303	1.684	2.021	2.423	2.704
11	1.363	1.796	2.201	2.718	3.106	**41**	1.303	1.683	2.020	2.421	2.701
12	1.356	1.782	2.179	2.681	3.055	**42**	1.302	1.682	2.018	2.418	2.698
13	1.350	1.771	2.160	2.650	3.012	**43**	1.302	1.681	2.017	2.416	2.695
14	1.345	1.761	2.145	2.624	2.977	**44**	1.301	1.680	2.015	2.414	2.692
15	1.341	1.753	2.131	2.602	2.947	**45**	1.301	1.679	2.014	2.412	2.690
16	1.337	1.746	2.120	2.583	2.921	**46**	1.300	1.679	2.013	2.410	2.687
17	1.333	1.740	2.110	2.567	2.898	**47**	1.300	1.678	2.012	2.408	2.685
18	1.330	1.734	2.101	2.552	2.878	**48**	1.299	1.677	2.011	2.407	2.682
19	1.328	1.729	2.093	2.539	2.861	**49**	1.299	1.677	2.010	2.405	2.680
20	1.325	1.725	2.086	2.528	2.845	**50**	1.299	1.676	2.009	2.403	2.678
21	1.323	1.721	2.080	2.518	2.831	**60**	1.296	1.671	2.000	2.390	2.660
22	1.321	1.717	2.074	2.508	2.819	**70**	1.294	1.667	1.994	2.381	2.648
23	1.319	1.714	2.069	2.500	2.807	**80**	1.292	1.664	1.990	2.374	2.639
24	1.318	1.711	2.064	2.492	2.797	**90**	1.291	1.662	1.987	2.368	2.632
25	1.316	1.708	2.060	2.485	2.787	**100**	1.290	1.660	1.984	2.364	2.626
26	1.315	1.706	2.056	2.479	2.779	**110**	1.289	1.659	1.982	2.361	2.621
27	1.314	1.703	2.052	2.473	2.771	**120**	1.289	1.658	1.980	2.358	2.617
28	1.313	1.701	2.048	2.467	2.763	**200**	1.286	1.653	1.972	2.345	2.601
29	1.311	1.699	2.045	2.462	2.756	∞	1.282	1.645	1.960	2.326	2.576
30	1.310	1.697	2.042	2.457	2.750						

To find a critical *t*-value, enter the table with df and a specified value for α, the significance level. For example, with 5 df, α = 0.05 and a one-tailed test, the desired probability in the tail would be $p = 0.05$ and the critical *t*-value would be $t(5, 0.05) = 2.015$. With α =0.05 and a two-tailed test, the desired probability in each tail would be $p = 0.025 = \alpha/2$, giving $t(0.025) = 2.571$. *Table generated using Excel.*

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Appendix C
Values of χ^2 (Degrees of Freedom, Level of Significance)

Degrees of Freedom	Probability in Right Tail								
	0.99	0.975	0.95	0.9	0.1	0.05	0.025	0.01	0.005
1	0.000157	0.000982	0.003932	0.0158	2.706	3.841	5.024	6.635	7.879
2	0.020100	0.050636	0.102586	0.2107	4.605	5.991	7.378	9.210	10.597
3	0.1148	0.2158	0.3518	0.5844	6.251	7.815	9.348	11.345	12.838
4	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.554	0.831	1.145	1.610	9.236	11.070	12.832	15.086	16.750
6	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.647	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	4.107	5.009	5.892	7.041	19.812	22.362	24.736	27.688	29.819
14	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289	42.796
23	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.558
25	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	12.878	14.573	16.151	18.114	36.741	40.113	43.195	46.963	49.645
28	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.994
29	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.335
30	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
50	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
80	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	116.321
100	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	140.170

To have a probability of 0.05 in the right tail when df = 5, the tabled value is $\chi^2(5, 0.05) = 11.070$.

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Appendix D
Table of the *F*-Distribution

Panel A. Critical values for right-hand tail area equal to 0.05

df2: \ df1:	1	2	3	4	5	6	7	8	9	10	11	12	15	20	21	22	23	24	25	30	40	60	120	∞
1	161	200	216	225	230	234	237	239	241	242	243	244	246	248	248	249	249	249	249	250	251	252	253	254
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.4	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5	19.5
3	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.76	8.74	8.70	8.66	8.65	8.65	8.64	8.64	8.63	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.94	5.91	5.86	5.80	5.79	5.79	5.78	5.77	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.70	4.68	4.62	4.56	4.55	4.54	4.53	4.53	4.52	4.50	4.46	4.43	4.40	4.37
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.03	4.00	3.94	3.87	3.86	3.86	3.85	3.84	3.83	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.60	3.57	3.51	3.44	3.43	3.43	3.42	3.41	3.40	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.31	3.28	3.22	3.15	3.14	3.13	3.12	3.12	3.11	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.10	3.07	3.01	2.94	2.93	2.92	2.91	2.90	2.89	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.94	2.91	2.85	2.77	2.76	2.75	2.75	2.74	2.73	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.82	2.79	2.72	2.65	2.64	2.63	2.62	2.61	2.60	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.72	2.69	2.62	2.54	2.53	2.52	2.51	2.51	2.50	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.63	2.60	2.53	2.46	2.45	2.44	2.43	2.42	2.41	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.57	2.53	2.46	2.39	2.38	2.37	2.36	2.35	2.34	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.51	2.48	2.40	2.33	2.32	2.31	2.30	2.29	2.28	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.46	2.42	2.35	2.28	2.26	2.25	2.24	2.24	2.23	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.41	2.38	2.31	2.23	2.22	2.21	2.20	2.19	2.18	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.37	2.34	2.27	2.19	2.18	2.17	2.16	2.15	2.14	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.34	2.31	2.23	2.16	2.14	2.13	2.12	2.11	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.31	2.28	2.20	2.12	2.11	2.10	2.09	2.08	2.07	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.28	2.25	2.18	2.10	2.08	2.07	2.06	2.05	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.26	2.23	2.15	2.07	2.06	2.05	2.04	2.03	2.02	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.24	2.20	2.13	2.05	2.04	2.02	2.01	2.01	2.00	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.22	2.18	2.11	2.03	2.01	2.00	1.99	1.98	1.97	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.20	2.16	2.09	2.01	2.00	1.98	1.97	1.96	1.96	1.92	1.87	1.82	1.77	1.71
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.13	2.09	2.01	1.93	1.92	1.91	1.90	1.89	1.88	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.04	2.00	1.92	1.84	1.83	1.81	1.80	1.79	1.78	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.95	1.92	1.84	1.75	1.73	1.72	1.71	1.70	1.69	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.87	1.83	1.75	1.66	1.64	1.63	1.62	1.61	1.60	1.55	1.50	1.43	1.35	1.25
Infinity	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.79	1.75	1.67	1.57	1.56	1.54	1.53	1.52	1.51	1.46	1.39	1.32	1.22	1.00

Numerator: df_1 and Denominator: df_2

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Appendix D (continued)
Table of the *F*-Distribution

Panel B. Critical values for right-hand tail area equal to 0.025

df2:	df1: 1	2	3	4	5	6	7	8	9	10	11	12	15	20	21	22	23	24	25	30	40	60	120	∞
1	648	799	864	900	922	937	948	957	963	969	973	977	985	993	994	995	996	997	998	1001	1006	1010	1014	1018
2	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.41	39.41	39.43	39.45	39.45	39.45	39.45	39.46	39.46	39.46	39.47	39.48	39.49	39.50
3	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.37	14.34	14.25	14.17	14.16	14.14	14.13	14.12	14.12	14.08	14.04	13.99	13.95	13.90
4	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.79	8.75	8.66	8.56	8.55	8.53	8.52	8.51	8.50	8.46	8.41	8.36	8.31	8.26
5	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.57	6.52	6.43	6.33	6.31	6.30	6.29	6.28	6.27	6.23	6.18	6.12	6.07	6.02
6	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.41	5.37	5.27	5.17	5.15	5.14	5.13	5.12	5.11	5.07	5.01	4.96	4.90	4.85
7	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.71	4.67	4.57	4.47	4.45	4.44	4.43	4.41	4.40	4.36	4.31	4.25	4.20	4.14
8	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.24	4.20	4.10	4.00	3.98	3.97	3.96	3.95	3.94	3.89	3.84	3.78	3.73	3.67
9	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.91	3.87	3.77	3.67	3.65	3.64	3.63	3.61	3.60	3.56	3.51	3.45	3.39	3.33
10	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.66	3.62	3.52	3.42	3.40	3.39	3.38	3.37	3.35	3.31	3.26	3.20	3.14	3.08
11	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.47	3.43	3.33	3.23	3.21	3.20	3.18	3.17	3.16	3.12	3.06	3.00	2.94	2.88
12	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.32	3.28	3.18	3.07	3.06	3.04	3.03	3.02	3.01	2.96	2.91	2.85	2.79	2.72
13	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	3.20	3.15	3.05	2.95	2.93	2.92	2.91	2.89	2.88	2.84	2.78	2.72	2.66	2.60
14	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	3.09	3.05	2.95	2.84	2.83	2.81	2.80	2.79	2.78	2.73	2.67	2.61	2.55	2.49
15	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	3.01	2.96	2.86	2.76	2.74	2.73	2.71	2.70	2.69	2.64	2.59	2.52	2.46	2.40
16	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05	2.99	2.93	2.89	2.79	2.68	2.67	2.65	2.64	2.63	2.61	2.57	2.51	2.45	2.38	2.32
17	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98	2.92	2.87	2.82	2.72	2.62	2.60	2.59	2.57	2.56	2.55	2.50	2.44	2.38	2.32	2.25
18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.81	2.77	2.67	2.56	2.54	2.53	2.52	2.50	2.49	2.44	2.38	2.32	2.26	2.19
19	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.76	2.72	2.62	2.51	2.49	2.48	2.46	2.45	2.44	2.39	2.33	2.27	2.20	2.13
20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.72	2.68	2.57	2.46	2.45	2.43	2.42	2.41	2.40	2.35	2.29	2.22	2.16	2.09
21	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80	2.73	2.68	2.64	2.53	2.42	2.41	2.39	2.38	2.37	2.36	2.31	2.25	2.18	2.11	2.04
22	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.65	2.60	2.50	2.39	2.37	2.36	2.34	2.33	2.32	2.27	2.21	2.14	2.08	2.00
23	5.75	4.35	3.75	3.41	3.18	3.02	2.90	2.81	2.73	2.67	2.62	2.57	2.47	2.36	2.34	2.33	2.31	2.30	2.29	2.24	2.18	2.11	2.04	1.97
24	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.59	2.54	2.44	2.33	2.31	2.30	2.28	2.27	2.26	2.21	2.15	2.08	2.01	1.94
25	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68	2.61	2.56	2.51	2.41	2.30	2.28	2.27	2.26	2.24	2.23	2.18	2.12	2.05	1.98	1.91
30	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.46	2.41	2.31	2.20	2.18	2.16	2.15	2.14	2.12	2.07	2.01	1.94	1.87	1.79
40	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.33	2.29	2.18	2.07	2.05	2.03	2.02	2.01	1.99	1.94	1.88	1.80	1.72	1.64
60	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.22	2.17	2.06	1.94	1.93	1.91	1.90	1.88	1.87	1.82	1.74	1.67	1.58	1.48
120	5.15	3.80	3.23	2.89	2.67	2.52	2.39	2.30	2.22	2.16	2.10	2.05	1.94	1.82	1.81	1.79	1.77	1.76	1.75	1.69	1.61	1.53	1.43	1.31
Infinity	5.02	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.11	2.05	1.99	1.94	1.83	1.71	1.69	1.67	1.66	1.64	1.63	1.57	1.48	1.39	1.27	1.00

Numerator: df_1 and Denominator: df_2

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Appendix D (continued)
Table of the *F*-Distribution

Panel C. Critical values for right-hand tail area equal to 0.01

		Numerator: df_1 and Denominator: df_2																						
	df1: 1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**15**	**20**	**21**	**22**	**23**	**24**	**25**	**30**	**40**	**60**	**120**	**∞**
df2: 1	4052	5000	5403	5625	5764	5859	5928	5982	6023	6056	6083	6106	6157	6209	6216	6223	6229	6235	6240	6261	6287	6313	6339	6366
2	98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4	99.4	99.4	99.4	99.4	99.4	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5	99.5
3	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3	27.2	27.1	27.1	26.9	26.7	26.7	26.6	26.6	26.6	26.6	26.5	26.4	26.3	26.2	26.1
4	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7	14.5	14.5	14.4	14.2	14.0	14.0	14.0	13.9	13.9	13.9	13.8	13.7	13.7	13.6	13.5
5	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2	10.1	10.0	9.89	9.72	9.55	9.53	9.51	9.49	9.47	9.45	9.38	9.29	9.20	9.11	9.02
6	13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.79	7.72	7.56	7.40	7.37	7.35	7.33	7.31	7.30	7.23	7.14	7.06	6.97	6.88
7	12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.54	6.47	6.31	6.16	6.13	6.11	6.09	6.07	6.06	5.99	5.91	5.82	5.74	5.65
8	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.73	5.67	5.52	5.36	5.34	5.32	5.30	5.28	5.26	5.20	5.12	5.03	4.95	4.86
9	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.18	5.11	4.96	4.81	4.79	4.77	4.75	4.73	4.71	4.65	4.57	4.48	4.40	4.31
10	10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.77	4.71	4.56	4.41	4.38	4.36	4.34	4.33	4.31	4.25	4.17	4.08	4.00	3.91
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.46	4.40	4.25	4.10	4.08	4.06	4.04	4.02	4.01	3.94	3.86	3.78	3.69	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.22	4.16	4.01	3.86	3.84	3.82	3.80	3.78	3.76	3.70	3.62	3.54	3.45	3.36
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	4.02	3.96	3.82	3.66	3.64	3.62	3.60	3.59	3.57	3.51	3.43	3.34	3.25	3.17
14	8.86	6.51	5.56	5.04	4.70	4.46	4.28	4.14	4.03	3.94	3.86	3.80	3.66	3.51	3.48	3.46	3.44	3.43	3.41	3.35	3.27	3.18	3.09	3.00
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.73	3.67	3.52	3.37	3.35	3.33	3.31	3.29	3.28	3.21	3.13	3.05	2.96	2.87
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.62	3.55	3.41	3.26	3.24	3.22	3.20	3.18	3.16	3.10	3.02	2.93	2.84	2.75
17	8.40	6.11	5.19	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.52	3.46	3.31	3.16	3.14	3.12	3.10	3.08	3.07	3.00	2.92	2.83	2.75	2.65
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.43	3.37	3.23	3.08	3.05	3.03	3.02	3.00	2.98	2.92	2.84	2.75	2.66	2.57
19	8.19	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.36	3.30	3.15	3.00	2.98	2.96	2.94	2.92	2.91	2.84	2.76	2.67	2.58	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.29	3.23	3.09	2.94	2.92	2.90	2.88	2.86	2.84	2.78	2.69	2.61	2.52	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.24	3.17	3.03	2.88	2.86	2.84	2.82	2.80	2.79	2.72	2.64	2.55	2.46	2.36
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.18	3.12	2.98	2.83	2.81	2.78	2.77	2.75	2.73	2.67	2.58	2.50	2.40	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.14	3.07	2.93	2.78	2.76	2.74	2.72	2.70	2.69	2.62	2.54	2.45	2.35	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.09	3.03	2.89	2.74	2.72	2.70	2.68	2.66	2.64	2.58	2.49	2.40	2.31	2.21
25	7.77	5.57	4.68	4.18	3.86	3.63	3.46	3.32	3.22	3.13	3.06	2.99	2.85	2.70	2.68	2.66	2.64	2.62	2.60	2.53	2.45	2.36	2.27	2.17
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.91	2.84	2.70	2.55	2.53	2.51	2.49	2.47	2.45	2.39	2.30	2.21	2.11	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.73	2.66	2.52	2.37	2.35	2.33	2.31	2.29	2.27	2.20	2.11	2.02	1.92	1.80
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.56	2.50	2.35	2.20	2.17	2.15	2.13	2.12	2.10	2.03	1.94	1.84	1.73	1.60
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.40	2.34	2.19	2.03	2.01	1.99	1.97	1.95	1.93	1.86	1.76	1.66	1.53	1.38
Infinity	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.25	2.18	2.04	1.88	1.85	1.83	1.81	1.79	1.77	1.70	1.59	1.47	1.32	1.00

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Appendix D (continued) Table of the *F*-Distribution

Panel D. Critical values for right-hand tail area equal to 0.005

	Numerator: df_1 and Denominator: df_2																							
	df1: 1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**15**	**20**	**21**	**22**	**23**	**24**	**25**	**30**	**40**	**60**	**120**	**∞**
df2: 1	16211	20000	21615	22500	23056	23437	23715	23925	24091	24222	24334	24426	24630	24836	24863	24892	24915	24940	24959	25044	25146	25253	25359	25464
2	198.5	199.0	199.2	199.2	199.3	199.3	199.4	199.4	199.4	199.4	199.4	199.4	199.4	199.4	199.4	199.4	199.4	199.4	199.4	199.5	199.5	199.5	199.5	200
3	55.55	49.80	47.47	46.20	45.39	44.84	44.43	44.13	43.88	43.68	43.52	43.39	43.08	42.78	42.73	42.69	42.66	42.62	42.59	42.47	42.31	42.15	41.99	41.83
4	31.33	26.28	24.26	23.15	22.46	21.98	21.62	21.35	21.14	20.97	20.82	20.70	20.44	20.17	20.13	20.09	20.06	20.03	20.00	19.89	19.75	19.61	19.47	19.32
5	22.78	18.31	16.53	15.56	14.94	14.51	14.20	13.96	13.77	13.62	13.49	13.38	13.15	12.90	12.87	12.84	12.81	12.78	12.76	12.66	12.53	12.40	12.27	12.14
6	18.63	14.54	12.92	12.03	11.46	11.07	10.79	10.57	10.39	10.25	10.13	10.03	9.81	9.59	9.56	9.53	9.50	9.47	9.45	9.36	9.24	9.12	9.00	8.88
7	16.24	12.40	10.88	10.05	9.52	9.16	8.89	8.68	8.51	8.38	8.27	8.18	7.97	7.75	7.72	7.69	7.67	7.64	7.62	7.53	7.42	7.31	7.19	7.08
8	14.69	11.04	9.60	8.81	8.30	7.95	7.69	7.50	7.34	7.21	7.10	7.01	6.81	6.61	6.58	6.55	6.53	6.50	6.48	6.40	6.29	6.18	6.06	5.95
9	13.61	10.11	8.72	7.96	7.47	7.13	6.88	6.69	6.54	6.42	6.31	6.23	6.03	5.83	5.80	5.78	5.75	5.73	5.71	5.62	5.52	5.41	5.30	5.19
10	12.83	9.43	8.08	7.34	6.87	6.54	6.30	6.12	5.97	5.85	5.75	5.66	5.47	5.27	5.25	5.22	5.20	5.17	5.15	5.07	4.97	4.86	4.75	4.64
11	12.23	8.91	7.60	6.88	6.42	6.10	5.86	5.68	5.54	5.42	5.32	5.24	5.05	4.86	4.83	4.80	4.78	4.76	4.74	4.65	4.55	4.45	4.34	4.23
12	11.75	8.51	7.23	6.52	6.07	5.76	5.52	5.35	5.20	5.09	4.99	4.91	4.72	4.53	4.50	4.48	4.45	4.43	4.41	4.33	4.23	4.12	4.01	3.90
13	11.37	8.19	6.93	6.23	5.79	5.48	5.25	5.08	4.94	4.82	4.72	4.64	4.46	4.27	4.24	4.22	4.19	4.17	4.15	4.07	3.97	3.87	3.76	3.65
14	11.06	7.92	6.68	6.00	5.56	5.26	5.03	4.86	4.72	4.60	4.51	4.43	4.25	4.06	4.03	4.01	3.98	3.96	3.94	3.86	3.76	3.66	3.55	3.44
15	10.80	7.70	6.48	5.80	5.37	5.07	4.85	4.67	4.54	4.42	4.33	4.25	4.07	3.88	3.86	3.83	3.81	3.79	3.77	3.69	3.59	3.48	3.37	3.26
16	10.58	7.51	6.30	5.64	5.21	4.91	4.69	4.52	4.38	4.27	4.18	4.10	3.92	3.73	3.71	3.68	3.66	3.64	3.62	3.54	3.44	3.33	3.22	3.11
17	10.38	7.35	6.16	5.50	5.07	4.78	4.56	4.39	4.25	4.14	4.05	3.97	3.79	3.61	3.58	3.56	3.53	3.51	3.49	3.41	3.31	3.21	3.10	2.98
18	10.22	7.21	6.03	5.37	4.96	4.66	4.44	4.28	4.14	4.03	3.94	3.86	3.68	3.50	3.47	3.45	3.42	3.40	3.38	3.30	3.20	3.10	2.99	2.87
19	10.07	7.09	5.92	5.27	4.85	4.56	4.34	4.18	4.04	3.93	3.84	3.76	3.59	3.40	3.37	3.35	3.33	3.31	3.29	3.21	3.11	3.00	2.89	2.78
20	9.94	6.99	5.82	5.17	4.76	4.47	4.26	4.09	3.96	3.85	3.76	3.68	3.50	3.32	3.29	3.27	3.24	3.22	3.20	3.12	3.02	2.92	2.81	2.69
21	9.83	6.89	5.73	5.09	4.68	4.39	4.18	4.01	3.88	3.77	3.68	3.60	3.43	3.24	3.22	3.19	3.17	3.15	3.13	3.05	2.95	2.84	2.73	2.61
22	9.73	6.81	5.65	5.02	4.61	4.32	4.11	3.94	3.81	3.70	3.61	3.54	3.36	3.18	3.15	3.12	3.10	3.08	3.06	2.98	2.88	2.77	2.66	2.55
23	9.63	6.73	5.58	4.95	4.54	4.26	4.05	3.88	3.75	3.64	3.55	3.47	3.30	3.12	3.09	3.06	3.04	3.02	3.00	2.92	2.82	2.71	2.60	2.48
24	9.55	6.66	5.52	4.89	4.49	4.20	3.99	3.83	3.69	3.59	3.50	3.42	3.25	3.06	3.04	3.01	2.99	2.97	2.95	2.87	2.77	2.66	2.55	2.43
25	9.48	6.60	5.46	4.84	4.43	4.15	3.94	3.78	3.64	3.54	3.45	3.37	3.20	3.01	2.99	2.96	2.94	2.92	2.90	2.82	2.72	2.61	2.50	2.38
30	9.18	6.35	5.24	4.62	4.23	3.95	3.74	3.58	3.45	3.34	3.25	3.18	3.01	2.82	2.80	2.77	2.75	2.73	2.71	2.63	2.52	2.42	2.30	2.18
40	8.83	6.07	4.98	4.37	3.99	3.71	3.51	3.35	3.22	3.12	3.03	2.95	2.78	2.60	2.57	2.55	2.52	2.50	2.48	2.40	2.30	2.18	2.06	1.93
60	8.49	5.79	4.73	4.14	3.76	3.49	3.29	3.13	3.01	2.90	2.82	2.74	2.57	2.39	2.36	2.33	2.31	2.29	2.27	2.19	2.08	1.96	1.83	1.69
120	8.18	5.54	4.50	3.92	3.55	3.28	3.09	2.93	2.81	2.71	2.62	2.54	2.37	2.19	2.16	2.13	2.11	2.09	2.07	1.98	1.87	1.75	1.61	1.43
Infinity	7.88	5.30	4.28	3.72	3.35	3.09	2.90	2.74	2.62	2.52	2.43	2.36	2.19	2.00	1.97	1.95	1.92	1.90	1.88	1.79	1.67	1.53	1.36	1.00

With 1 degree of freedom (df) in the numerator and 3 df in the denominator, the critical *F*-value is 10.1 for a right-hand tail area equal to 0.05.

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Appendix E
Critical Values for the Durbin-Watson Statistic (α = .05)

	K = 1		*K* = 2		*K* = 3		*K* = 4		*K* = 5	
n	d_l	d_u	d_l	d_u	d_l	d_u	d_l	d_u	d_l	d_u
15	1.08	1.36	0.95	1.54	0.82	1.75	0.69	1.97	0.56	2.21
16	1.10	1.37	0.98	1.54	0.86	1.73	0.74	1.93	0.62	2.15
17	1.13	1.38	1.02	1.54	0.90	1.71	0.78	1.90	0.67	2.10
18	1.16	1.39	1.05	1.53	0.93	1.69	0.82	1.87	0.71	2.06
19	1.18	1.40	1.08	1.53	0.97	1.68	0.86	1.85	0.75	2.02
20	1.20	1.41	1.10	1.54	1.00	1.68	0.90	1.83	0.79	1.99
21	1.22	1.42	1.13	1.54	1.03	1.67	0.93	1.81	0.83	1.96
22	1.24	1.43	1.15	1.54	1.05	1.66	0.96	1.80	0.86	1.94
23	1.26	1.44	1.17	1.54	1.08	1.66	0.99	1.79	0.90	1.92
24	1.27	1.45	1.19	1.55	1.10	1.66	1.01	1.78	0.93	1.90
25	1.29	1.45	1.21	1.55	1.12	1.66	1.04	1.77	0.95	1.89
26	1.30	1.46	1.22	1.55	1.14	1.65	1.06	1.76	0.98	1.88
27	1.32	1.47	1.24	1.56	1.16	1.65	1.08	1.76	1.01	1.86
28	1.33	1.48	1.26	1.56	1.18	1.65	1.10	1.75	1.03	1.85
29	1.34	1.48	1.27	1.56	1.20	1.65	1.12	1.74	1.05	1.84
30	1.35	1.49	1.28	1.57	1.21	1.65	1.14	1.74	1.07	1.83
31	1.36	1.50	1.30	1.57	1.23	1.65	1.16	1.74	1.09	1.83
32	1.37	1.50	1.31	1.57	1.24	1.65	1.18	1.73	1.11	1.82
33	1.38	1.51	1.32	1.58	1.26	1.65	1.19	1.73	1.13	1.81
34	1.39	1.51	1.33	1.58	1.27	1.65	1.21	1.73	1.15	1.81
35	1.40	1.52	1.34	1.58	1.28	1.65	1.22	1.73	1.16	1.80
36	1.41	1.52	1.35	1.59	1.29	1.65	1.24	1.73	1.18	1.80
37	1.42	1.53	1.36	1.59	1.31	1.66	1.25	1.72	1.19	1.80
38	1.43	1.54	1.37	1.59	1.32	1.66	1.26	1.72	1.21	1.79
39	1.43	1.54	1.38	1.60	1.33	1.66	1.27	1.72	1.22	1.79
40	1.44	1.54	1.39	1.60	1.34	1.66	1.29	1.72	1.23	1.79
45	1.48	1.57	1.43	1.62	1.38	1.67	1.34	1.72	1.29	1.78
50	1.50	1.59	1.46	1.63	1.42	1.67	1.38	1.72	1.34	1.77
55	1.53	1.60	1.49	1.64	1.45	1.68	1.41	1.72	1.38	1.77
60	1.55	1.62	1.51	1.65	1.48	1.69	1.44	1.73	1.41	1.77
65	1.57	1.63	1.54	1.66	1.50	1.70	1.47	1.73	1.44	1.77
70	1.58	1.64	1.55	1.67	1.52	1.70	1.49	1.74	1.46	1.77
75	1.60	1.65	1.57	1.68	1.54	1.71	1.51	1.74	1.49	1.77
80	1.61	1.66	1.59	1.69	1.56	1.72	1.53	1.74	1.51	1.77
85	1.62	1.67	1.60	1.70	1.57	1.72	1.55	1.75	1.52	1.77
90	1.63	1.68	1.61	1.70	1.59	1.73	1.57	1.75	1.54	1.78
95	1.64	1.69	1.62	1.71	1.60	1.73	1.58	1.75	1.56	1.78
100	1.65	1.69	1.63	1.72	1.61	1.74	1.59	1.76	1.57	1.78

Note: *K* = the number of slope parameters in the model.

Source: From J. Durbin and G. S. Watson, "Testing for Serial Correlation in Least Squares Regression, II." *Biometrika* 38 (1951): 159–178.